Free products of completely positive maps

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March 366th, 2020

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- M acts by left multiplicaton on $L^2(M, \tau)$ in the GNS representation of τ
- $\hat{x} \in L^2(M, \tau)$ corresponds to $x \in M$
- $E_N : M \to N$ is the τ-preserving conditional expectation, with $e_N \in \mathcal{B}(L^2(M))$ the corresponding projection.

More setup

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- Set $T_\Phi(\hat{x}) = \widehat{\Phi(x)}$ for $x \in M$ and extend by continuity
- We can decompose

$$T = \begin{pmatrix} I & 0 \\ 0 & T^0 \end{pmatrix}$$

where we've written $L^2(M) = L^2(N) \oplus L^2(N)^\perp$.

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- Recall in the definition of the Haagerup property that T_ϕ must be a compact operator on $L^2(M)$.
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- Define $F_N(M) = \{T \in N' \cap \mathcal{B}(L^2(M)) \mid T = \sum_{i=1}^k a_i e_N b_i\}$
- Let $\mathcal{K}_N(M)$ be the norm closure of $F_N(M)$ in $\mathcal{B}(L^2(M))$

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1. $\lim_i \|\Phi_i(x) - x\|_2 = 0$ for $x \in M$
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- This definition differs slightly from the one in Popa's book. Here we have unital and E_N preserving. There we have subunital and subtracial.
- $T^0_{\Phi_i}$ can be assumed to be a contraction in the definition, by considering

$$\Phi_{i,\varepsilon} = \frac{1}{1+\varepsilon}(\Phi_i + \varepsilon E_N)$$

and seeing that this makes $T^0_{\Phi_{i,\varepsilon}}$ a contraction. We'll use this later.

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$$M_0^0 = N \oplus \bigoplus_{n \geq 1, i_1 \neq \ldots \neq i_n} M_{i_1}^0 \otimes_N \cdots \otimes_N M_{i_n}^0$$

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- Define the map $E_0 : M_0^0 \to N$ by

$$E_0(x) = \begin{cases} x & \text{for } x \in N \\ 0 & \text{for } x \text{ in the other summand} \end{cases}$$

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$$L^2(M) = L^2(N) \oplus \bigoplus_{n \geq 1, i_1 \neq \ldots \neq i_n} (L^2(M_{i_1}) \ominus L^2(N)) \otimes_N \cdots \cdots \otimes_N (L^2(M_{i_n}) \ominus L^2(N))$$

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Finally, E_0 extends to a τ-preserving conditional expectation $E : M \to N$ and M_0^0 is a weakly dense $*$-subalg of M.

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- **This is completely positive on M_0^0.**
  - this takes some work
- We can extend this to Φ cp on $M_1 *_N M_2$?
  - this also takes a bit of work, but not as much (see p219 of [2])

Why is Φ_0 cp?

We're going to show Φ_0 is cp on M_0^0 by directly finding the Stinespring dilation from the dilations of Φ_1 and Φ_2. We'll be using a pretty technical version where we understand the spaces better. We'll be writing $H = L^2M$, $H_i = L^2M_i$ for $i = 1, 2$, and

$$H_0 = N \oplus \bigoplus H_{i_1}^0 \otimes \cdots \otimes H_{i_n}^0$$

as the free product with identity $\xi = I_N \oplus 0$.

Stinespring dilations of Φ_i

- Viewing $\Phi_i : M_i \to \mathcal{B}(L^2 M) =: \mathcal{B}(H)$ for $i = 1, 2$, write

$$\Phi_i = V_i^* \rho_i V_i$$

where $\rho_i : \mathcal{B}(H) \to \mathcal{B}(K_i)$ is a unital representation and $V : H \to K_i$ is an inclusion. Also $K_i = \overline{\text{span}}(\rho_i)(M_i)H$.

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- We also have $\rho_i(N)K_i^0 \subseteq K_i^0$ (a bit of work too)
- Write $\rho_i^0 \restriction_{K_i^0} : N \to \mathcal{B}(K_i^0)$

Defining a new space, part 1

- Set $X_i = \bigoplus_{n \geq 1, i_1 \neq \ldots \neq i_n \neq i} H_{i_1}^0 \otimes_N \cdots \otimes_N H_{i_n}^0$ as an N-bimodule.
- Set Y_i to be the same except $i_1 \neq 1$.

Defining a new space

Define

$$K = H \oplus \left(\bigoplus_i K_i^0 \right) \oplus \left(\bigoplus_i X_i \otimes_{\rho_i^0} K_i^0 \right)$$
$$= \cdots = K_i \oplus \left(X_i \otimes_{\rho_i^0} K_i^0 \right) \oplus \bigoplus_{j \neq i} (N \oplus X_j) \otimes_{\rho_j^0} K_j^0$$

here $\otimes_{\rho_i^0}$ is the completion of the algebraic tensor product with the scalar product induced by ρ_i^0.

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here $\otimes_{\rho_i^0}$ is the completion of the algebraic tensor product with the scalar product induced by ρ_i^0.
We define $\tilde{\rho}_i : M_i \to \mathcal{B}(K)$ by

$$\tilde{\rho}_i(a) = \rho_i(a) \oplus \left(\sigma_i(a)\restriction_{\bigoplus X_i} \otimes 1_{K_i^0} \right) \oplus \bigoplus_{j \neq i} \sigma_{ij}(a)$$

in an effort to extend ρ_i.

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  - $\sigma_i = \lambda_i \pi_i$
  - $\lambda_i : \mathcal{B}(H_i) \to \mathcal{B}(H_0)$ is defined by

$$\lambda_i(T) = V_i^{-1}(T \otimes I)V_i$$

What's σ_{ij}?

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$$\sigma_{ij}(a) = (W_j \oplus I_j)\left(\sigma_i(a)\restriction_{N \oplus X_j \otimes I_{K_j^0}}\right)(W_j^* \oplus I_j)$$

Think of this as another way of extending ρ_i.

Are the $\tilde{\rho}_i$'s compatible? A ρ from the $\tilde{\rho}_i$'s!

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- If these "extensions" $\tilde{\rho}_i$ differ on N, we can't define a ρ on the free product $M_1 *_N M_2$.
- This is not obvious but follows from the way we set up σ_i and σ_{ij}.
- Finally, set $\rho = \tilde{\rho}_1 * \tilde{\rho}_2 : M_0^0 \to \mathcal{B}(K)$. We'd like to show this is the Stinespring dilation of Φ_0 as defined in the previous section. As a reminder:

$$\Phi_0(x) = \begin{cases} x & \text{for } x \in N \\ \Phi_{i_1}(a_1) \ldots \Phi_{i_n}(a_n) & \text{for } x = a_1 \ldots a_n \end{cases}$$

The consequence of this (and the goal) is that Φ_0 is cp

Comparing $\tilde{\rho}_i$ with Φ_i

- It's enough to show that for $h, h' \in H$ (the smaller space), $x \in M$, we have

$$\langle \rho(x)h, h' \rangle = \langle \Phi(x)h, h' \rangle$$

for then ρ will satisfy $\Phi = V^* \rho V$ for the inclusion $V : H \to K$.

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- Showing this for $x = n \in N$ follows from what ρ_i does on N (being dilations of Φ_i) and the definition of $\tilde{\rho}_i$.

Comparing $\tilde{\rho}_i$ with Φ_i (continued)

- Finally, for $a_j \in M^0_{i_j}$, $1 \leq j \leq n$, and $i_1 \neq \ldots \neq i_n$ we want to verify

$$\left\langle \tilde{\rho}_{i_1}(a_1) \ldots \tilde{\rho}_{i_n}(a_n)h, h' \right\rangle = \left\langle \Phi_{i_1}(a_1) \ldots \Phi_{i_n}(a_n)h, h' \right\rangle$$

Comparing $\tilde{\rho}_i$ with Φ_i (continued)

- Finally, for $a_j \in M_{i_j}^0$, $1 \leq j \leq n$, and $i_1 \neq \ldots \neq i_n$ we want to verify

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- This is done by induction. First, the base case:

$$\begin{aligned} \tilde{\rho}_{i_n}(a_n)h &= \rho_{i_n}(a_n)h \\ &= \Phi_{i_n}(a_n)h + \left(\rho_{i_n}(a_n) - \Phi_{i_n}(a_n)\right) h \\ &= \Phi_{i_n}(a_n)h + k_n \end{aligned}$$

where $k_n \in K_{i_n}^0$ since ρ_{i_n} is the Stinespring dilation of Φ_{i_n}.

Rest of base case

$$\begin{aligned}\tilde{\rho}_{i_{n-1}}(a_{n-1})\tilde{\rho}_{i_n}(a_n)h &= \tilde{\rho}_{i_{n-1}}(a_{n-1})\left(\Phi_{i_n}(a_n)h + k_n\right)\\ &= \Phi_{i_{n-1}}(a_{n-1})\Phi_{i_n}(a_n)h\\ &\quad + \left(\rho_{i_{n-1}}(a_{n-1}) - \Phi_{i_{n-1}}(a_{n-1})\right)\Phi_{i_n}(a_n)h\\ &\quad + \sigma_{i_{n-1}}(a_{n-1}) \otimes k_n\end{aligned}$$

In other words, we can write this as $\Phi_{i_{n-1}}(a_{n-1})\Phi_{i_n}(a_n)h + \eta_{n-2}$ where

$$\eta_{n-2} \in K^0_{i_{n-1}} \oplus \bigoplus_{s=n-1}^{n-1} (H^0_{i_{n-1}} \otimes \cdots \otimes H^0_{i_s}) \otimes K^0_{i_s}.$$

Of course here that big sum isn't interesting, but later it will have more terms.

Inductive step

Assuming

$$\tilde{\rho}_{i_{k+1}}(a_{k+1}) \dots \tilde{\rho}_{i_n}(a_n)h = \Phi_{i_{k+1}}(a_{k+1}) \dots \Phi_{i_n}(a_n)h + \eta_k$$

where

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we can show this holds for the kth term thrown on too.

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where

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we can show this holds for the kth term thrown on too. This follows from a quick calculation and the fact that $\tilde{\rho_{i_k}}(a_k)$ is $\Phi_{i_k}(a_k) + (\rho_{i_k}(a_k) - \Phi_{i_k}(a_k)$ as before.

Finishing up induction

- Finally, since the leftover term is perpendicular to H,

$$\left\langle \rho(a_1 \dots a_n)h, h' \right\rangle = \left\langle \Phi_{i_1}(a_1) \dots \Phi_{i_n}(a_n)h, h' \right\rangle$$

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- Note that we can actually get further that $K = \overline{\operatorname{span}}\, \pi(M)H$ where π is the GNS rep for M into L^2M.
- To wrap up: since ρ is the Stinespring dilation of Φ_0, we must have that Φ_0 is cp.

Lemma

Set $X_j^0 = \sum_{k_j \in F_j} a_{j_{k_j}} e_N b_{j_{k_j}} \in N' \cap B(L^2(M_j^0))$ with F_j finite sets and $a_{j_{k_j}}, b_{j_{k_j}} \in M_j$. Then

$$X_{i_1}^0 \otimes \cdots \otimes X_{i_n}^0 = \sum_{j=1}^{n} \sum_{k_j \in F_j} a_{i_1 k_{i_1}} \dots a_{i_n k_{i_n}} e_N b_{i_1 k_{i_n}} \dots b_{i_1 k_{i_1}} \restriction_{L^2(M_{i_1}^0) \otimes \cdots \otimes L^2(M_{i_n}^0)}$$

for all $i_1 \neq \dots i_n$, $n \geq 1$.
This says that (some) tensors of things in $F_N(M)$ will still be in $F_N(M)$, restricted to the right domain.

. . . relies on

If $a_j \in M_{i_j}$ and $b_j \in M_{i_j}^0$ for $1 \leq j \leq n$, $i_1 \neq \ldots \neq i_n$, then

$$E_N(a_n \ldots a_1 b_1 \ldots b_n) = E_N(a_n \ldots a_2 E_N(a_1 b_1) b_2 \ldots b_n)$$

Statement

If $M_1, M_2 \supseteq N$ both have property (H) relative to N, then $M = M_1 *_N M_2$ has property (H) relative to N, with respect to $\tau_{M_1} * \tau_{M_2}$.

Proof setup

- Select $\Phi_{1,i}$ and $\Phi_{2,i}$ nets of cp maps as in the definition of (H)
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Proof setup

- Select $\Phi_{1,i}$ and $\Phi_{2,i}$ nets of cp maps as in the definition of (H)
  - Assume same index set I by taking a product net
- $\| T^0_{\Phi_{1,i}} \|, \| T^0_{\Phi_{2,i}} \|$ can be assumed both strict contractions
  - $\max = \rho_i < 1$
  - $T = \begin{pmatrix} I & 0 \\ 0 & T^0 \end{pmatrix}$ where $L^2 M_i = L^2 N \oplus L^2 M_i^0$.

Using the free product of cp maps

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- It's an E_N-preserving, N-bimodular, unital cp map
- We can decompose

$$\begin{aligned} T_{\Phi_i} &= T_{\Phi_{1,i}} * T_{\Phi_{2,i}} \\ &= I_{L^2 N} \oplus \bigoplus T^0_{\Phi_{j_1,i}} \otimes \cdots \otimes T^0_{\Phi_{j_n,i}} \end{aligned}$$

Verifying the limit condition

- Each $\|T_{\Phi_i}\| \leq 1$ so we just need to check $\lim_i \|\Phi_i(x) - x\|_2 = 0$ for $x \in M$ on finite sums of reduced words, since the tail will be irrelevant

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- Actually, since in $\mathrm{L}^2(\mathrm{M})$ different type words are orthogonal, just need reduced words x
- But remember for reduced words $a_1 \dots a_n$, Φ_i is the product of the $\Phi_{j,i}$'s

Verifying $T_{\Phi_i} \in \mathcal{K}_N(M)$

Fix i, and just write $T_j = T_{\Phi_{j,i}}$.

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Denote

$$X_j^0 = (1 - e_N)X_j(1 - e_N).$$

This is consistent notation! X_j^0 acts on $L^2(M_j^0)$, still is a strict contraction, and is still ε close to T_j^0.

Verifying $T_{\Phi_i} \in \mathcal{K}_N(M)$, calculation

$$
\begin{aligned}
\|T^0_{k_1} \otimes \cdots \otimes T^0_{k_n} - X^0_{k_1} \otimes \cdots \otimes X^0_{k_n}\| \leq{} & \|T^0_{k_1} - X^0_{k_1}\| \|T^0_{k_2}\| \dots \|T^0_{k_n}\| \\
& + \|X^0_{k_1}\| \|T^0_{k_2} - X^0_{k_2}\| \|T^0_{k_3}\| \dots \|T^0_{k_n}\| \\
& + \dots \\
& + \|X^0_{k_1}\| \dots \|X^0_{k_{n-1}}\| \|T^0_{k_n} - X^0_{k_n}\|
\end{aligned}
$$

by the triangle inequality.

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by the triangle inequality.

- X^0's are contractions, so this is bounded by $\varepsilon(\rho_i^{n-1} + \cdots + \rho_i + 1) \leq \varepsilon/(1 - \rho_i)$ (also < 1).

Wrapping up

- Lemma: since the X^0's are in $N' \cap L^2(M_j^0)$, taking all terms up to order m gives us

$$I_{L^2N} \oplus \bigoplus_{n \leq m, k_1 \neq \ldots \neq k_n} X^0_{k_1} \otimes \cdots \otimes X^0_{k_n} \in F_N(M)$$

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- Since T_{Φ_i} is approximated in norm by members of $F_N(M)$, it must be in $\mathcal{K}_N(M)$, and the second condition is met.
- Thus M has property (H) relative to N and the witnessing cp maps are the free products with amalgamations of the cp maps for M_1 and M_2.

References

1. Florin Boca. "On the method of constructing irreducible finite index subfactors of Popa." Pacific J. Math. 161 (2) 201 - 231, 1993. Link
   - It's basically just section 3 up to proposition 3.9
2. Florin Boca. "Completely positive maps on amalgamated product C*-algebras." Math. Scandinavica. 72, 212-222, 1993. Link
   - This contains the proof that Φ_0 is cp and can be extended to Φ