Amenable Groups

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  - reminder: $\mu(1) = 1$ and μ is positive
- Assume every group is discrete from here on out.

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2. We could have equivalently defined G as amenable if there is a left-invariant finitely additive probability measure.
   - given a finitely additive measure m, the integral $\int \cdot \, dm$ is our invariant mean
   - given an invariant mean μ, $m(A) = \mu(\chi_A)$ is our invariant finitely additive probability measure

Quick examples

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- Extending our definition to locally compact groups, compact groups are amenable. The Haar measure is our left invariant mean (in the sense of measure).
- $\mathbb{Z}^n$ is amenable, and in fact every abelian group is amenable.

Følner Sequences

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- Informally, large F_n's don't move much when pushed by any fixed element of G.
- Equivalent to amenability

An informal example

$\mathbb{Z}^n$ has a Følner sequence given by $F_m = \{(z_1, \ldots, z_m) \mid |z_i| \leq m\}$.

- after perturbing this set by any element $g \in \mathbb{Z}^n$, we see that only F_m's "boundary" gets counted by $|gF_m \triangle F_m|$, and the surface area of a box is small relative to the volume for large boxes.

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or:

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or:

- push a square just a bit: the leftovers are linear but the area is quadratic so the ratio goes to zero

Følner's Theorem

A discrete *countable* group G has a Følner sequence iff it is amemable.

A first attempt at the forward direction

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- Example: consider $F_n = [-n, n]$ a Følner sequence for $\mathbb{Z}$. This definition yields the asymptotic density of A in $\mathbb{Z}$.
- Problem: does the limit exist (and define something reasonable?)

If G has a Følner sequence, it is amenable

- Let F_n be a Følner sequence for G and define a finitely additive probability measure on G by

$$\mu(A) = \lim_{\omega} \frac{|F_n \cap A|}{|F_n|}$$

where ω is a nonprincipal ultrafilter on $\mathbb{N}$.

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where ω is a nonprincipal ultrafilter on $\mathbb{N}$.
- $\mu(G) = 1$ because the limit exists and ultralimits agree with limits
- We have left invariance
$$\begin{aligned} |\mu(gA) - \mu(A)| &\leq \frac{1}{|F_n|}\big||F_n \cap gA| - |F_n \cap A|\big| \\ &\leq \frac{1}{|F_n|}\left|(g^{-1}F_n \triangle F_n) \cap A\right| \to 0 \end{aligned}$$

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Let's show now that countable discrete amenable groups have Følner sequences. We'll do this in a few steps using an argument of Namioka:
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2. Given a finite set S and $\varepsilon > 0$, we can find a large F such that

$$\frac{|gF \triangle F|}{|F|} \leq \varepsilon$$

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2. Given a finite set S and $\varepsilon > 0$, we can find a large F such that

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   for each $g \in S$
3. The existence of a Følner sequence follows.

Følner's Theorem: Step 1

Suppose G is an amenable countable discrete group. Let $\mu \in \ell^\infty(G)$ be a left invariant mean. We'd like to show that for all finite sets $S \subseteq G$ and $\varepsilon > 0$, there exists a *finite mean*, i.e., a finitely supported function $\nu : G \to \mathbb{R}^+$ with $\|\nu\|_{\ell^1(G)} = 1$ satisfying:

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Suppose not. Then there exists $S \subseteq G$ finite and $\varepsilon > 0$ with $\sup_{g\in S}\|\nu - L_g\nu\|_{\ell^1(G)} \geq \varepsilon$ for every finite mean ν.

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- Taking the norm $\sup_{g \in S} \|f(g, \cdot)\|$ for the space $(\ell^1(G))^S$, note that $\{(\nu - L_g \nu)_{g \in S} \mid \nu \in V\}$ is a convex subset of $V^S \subseteq (\ell^1(G))^S$ bounded away from zero.

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- Hahn-Banach separation yields a functional $\alpha \in ((\ell^1(G))^S)^*$, with

$$\alpha_g(\nu - L_g\nu) > 1$$

for all $\nu \in V$ and $g \in S$ where we write α as $(\alpha_g)_{g\in S}$.

Følner's Theorem: Step 1

- Rewriting our previous condition, using the fact that $((\ell^1(G))^S)^* \simeq (\ell^\infty(G))^S$, we get for each $g \in S$ a function $\beta_g \in \ell^\infty(G)$ satisfying

$$\sum_x \beta_g(x)\,(\nu(x) - (L_g\nu)(x)) > 1$$

for all finite means ν.

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for all finite means ν.

- Let's consider $\nu = \delta_h$ to get

$$\beta_g(h) - (L_{g^{-1}}\beta_g)(h) > 1$$

which holds for every $h \in G$.

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a contradiction to left-invariance.
Thus given our invariant mean μ, it's true that for all S finite and $\varepsilon > 0$, there's a finite mean ν satisfying $\forall g \in S$,

$$\|\nu - L_g\nu\|_{\ell^1(G)} < \varepsilon$$

Følner's Theorem: Step 2

Goal: for every S finite and $\varepsilon > 0$ there exists F finite satisfying

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Let S be finite and $\varepsilon > 0$. By Step 1, we have a finite mean ν with

$$\|\nu - L_g \nu\|_{\ell^1(G)} \leq \frac{\varepsilon}{|S|}$$

for all $g \in S$.

Følner's Theorem: Step 2

Since ν is finitely supported, let's take its "layer cake decomposition":

$$\nu = \sum_{i=1}^{n} c_i \chi_{F_i}$$

with $c_i > 0$ and $F_1 \supseteq \cdots \supseteq F_n$.

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$$\nu = \sum_{i=1}^{n} c_i \chi_{F_i}$$

with $c_i > 0$ and $F_1 \supseteq \cdots \supseteq F_n$. Note that $\sum c_i |F_i| = 1$ since ν is a finite mean.

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Note that $|\nu(g) - (L_h\nu)(g)| \geq c_i$ for $g \in hF_i \triangle F_i$. Think of hopping up or down a layer of the cake. Integrating the above,

$$\sum_{i=1}^{n} c_i |hF_i \triangle F_i| \leq \|\nu - L_h\nu\|_{\ell^1(G)} \leq \frac{\varepsilon}{|S|} \sum_{i=1}^{n} c_i |F_i|$$

using $\sum c_i |F_i| = 1$.

Følner's Theorem: Step 2

Summing in $h \in S$,

$$\sum_{i=1}^{n} \sum_{h \in S} c_i |hF_i \triangle F_i| \leq \varepsilon \sum_{i=1}^{n} c_i |F_i|$$

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$$\sum_{i=1}^{n} \sum_{h \in S} c_i |hF_i \triangle F_i| \leq \varepsilon \sum_{i=1}^{n} c_i |F_i|$$

By pigeonhole, there must exist i with

$$\sum_{h \in S} \frac{|hF_i \triangle F_i|}{|F_i|} \leq \varepsilon$$

This F_i is our desired set.

Følner's Theorem: Step 3

Now we know that for all S and $\varepsilon > 0$ there exists F satisfying $|gF \triangle F|/|F| \leq \varepsilon$ for all $g \in S$, we'll find a Følner sequence. For each $\varepsilon = 1/n$ select F_n satisfying the above.

Quotients of amenable groups are amenable

Let $H = G/N$ be a quotient of an amenable discrete group. Given μ on G, define ν on H by

$$\nu(A) = \mu(AN)$$

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- Define $f_A : G \to \mathbb{R}$ via $f_A(g) = \mu(N \cap g^{-1}A)$ for $A \subseteq G$.

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- Pull back to $f_A : G/N \to \mathbb{R}$, noting μ is N invariant.

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- Pull back to $f_A : G/N \to \mathbb{R}$, noting μ is N invariant.
- Define $\psi(A) = \int f_A(x)\, d\nu(x)$.

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Let H be a subgroup of G (discrete). By the axiom of choice, let S contain precisely one element of every right coset of H. Given a probability measure μ on G, define ν on H via $\nu(A) = \mu(AS)$.

Discrete abelian groups are amenable

This fact follows from a few things:

- Direct limits of amenable groups are amenable
- Direct sums of amenable groups are amenable

Free groups (except $\mathbb{Z}$) are not amenable

- Picture the Cayley graph of $\mathbb{F}_2$. Any large set has very large "boundary", so informally we should be concerned about the existence of a Følner sequence.
- Since subgroups of amenable groups are amenable, it suffices to prove that $\mathbb{F}_2$ is not amenable.

A Banach-Tarski Trick

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1. $\mu(W(a)) + \mu(W(a^{-1})) = 1$, and so $\mu(W(b)) = \mu(W(b^{-1})) = 0$.

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2. Similarly for b.

Sources

Mostly

- http://reh.math.uni-duesseldorf.de/~garrido/amenable.pdf
- https://terrytao.wordpress.com/2009/04/14/some-notes-on-amenability/