Qualifying Exam Solutions

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Note almost all of these are taken from Adam Lott's typed solutions, which are much cleaner than these solutions. Several are also shamelessly ripped from the grad wiki. I also received some help from some friends on these.

These were typed up for my own reference and include a few random thoughts in addition to the problem. It's messy, there may be occasional typos, and no guarantee is made about accuracy, but do let me know if there are mathematical errors.

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Problem 1

Some of the following statements about sequences of functions f_n in $L^3([0,1])$ are false. Indicate these and provide an appropriate counter-example.

1. If $f_n \to f$ ae then a subsequence converges to f in L^3
2. If $f_n \to f$ in L^3 then a subsequence converges ae.
3. If $f_n \to f$ in measure (i.e. probability) then the sequences converges to f in L^3
4. If $f_n \to f$ in L^3 then the sequence converges in measure.

Proof

1. This is false. Consider a bump moving off to ∞, i.e., $\chi_{[n,n+1]}$.
2. This is true.
3. This is false. Consider $\frac{1}{n^{1/3}}\chi_{n,2n}$. These are all norm 1 in L^3, but $m\{f_n > \varepsilon\} \to 0$ for every ε.
4. This is true.

Problem 2

Let X and Y be topological spaces and $X \times Y$ the Cartesian product endowed with the product topology. $B(X)$ denotes the Borel sets in X, etc.

1. Suppose $f : X \to Y$ is continuous. Prove that $E \in B(Y)$ implies $f^{-1}(E) \in B(X)$.

2. Suppose that $A \in B(X)$ and $E \in B(Y)$. Show that $A \times E \in B(X \times Y)$.

Proof, Part 1

Let's consider $\{E \in B(Y) \mid f^{-1}(E) \text{ is Borel}\}$. This contains all the open sets, is closed under complements and countable unions, and thus is $B(Y)$.

Proof, Part 2

We write $A \times E = \pi_1^{-1}(A) \cap \pi_2^{-1}(E)$.

Problem 3

Given $f : [0, 1] \to \mathbb{R}$ in L^1, define

$$f_n(x) = n \int_{k/n}^{(k+1)/n} f(y)\,dy \text{ for } y \in [k/n, (k+1)/n), k = 0, \ldots, n-1$$

Prove that $f_n \to f$ in L^1.

Proof

Let's prove this for characteristic functions first. In particular, it's easy to see that it's true for intervals or finite unions of intervals.

Such functions are L^1 dense in characteristic functions. Finite linear combinations don't affect the result, and so we can consider simple functions, which are L^1 dense in all L^1 functions.

Thus if the result holds when passing to L^1 limits, we're done. Take $\|g - f\|_{L^1} < \varepsilon/3$ where the result holds for g. Then take $N \in \mathbb{N}$ such that $\|g_n - g\|_{L^1} < \varepsilon/3$. Finally, note that $\|f_k - g_k\| \leq \|f - g\|$, so $\|f_n - f\|_{L^1} \leq \varepsilon$ for all $n \geq N$.

Problem 4

Let $S = \{f \in L^1(\mathbb{R}^3) \mid \int f\,dx = 0\}$.

1. Show that S is closed in the L^1 topology
2. Show that $S \cap L^2$ is dense in L^2

Proof, Part 1

Let $f_n \to f$ in L^1. Then $\int f \leq \liminf \int f_n = 0$ by Fatou. Same goes for $-f$ and we're done.

Alternatively, $\left|\int f\right| = \left|\int f - \int f_n\right| \leq \int |f - f_n| \to 0$.

Proof, Part 2

It's enough to approximate a member of $L^2([-R,R])$ where R may be arbitrary.

Let $f \in L^2([-R,R])$. Define $g = f$ on $[-R,R]$. Let $\varepsilon > 0$. On $[R, R+\alpha]$, let g take the constant value $\frac{-1}{\alpha}\int_{-R}^{R} f =: \frac{-1}{\alpha} I$.

Clearly $g \in S$, but now $\|f-g\|_{L^2} = I^2/\alpha$. By making α large, this can be made less than ε as desired.

I solved this assuming the domain was $\mathbb{R}$ accidentally. There is basically no difference for $\mathbb{R}^3$, except use an annulus instead of an interval.

Problem 5

State and prove the Riesz representation theorem for linear functionals on a (separable) Hilbert space.

Proof

Let $L : H \to \mathbb{R}$ be a continuous linear functional on a separable Hilbert space. Then there exists some $v \in H$ such that $L(x) = \langle x, v\rangle$.

Note that L has a closed kernel of codimension 1. Let $u \in \ker(L)^\perp$. I claim that $L(x) = \alpha\langle x, u\rangle$ for some α. Select $\alpha = \overline{L(u)}/\|u\|^2$ and set $v = \alpha u$.

It suffices to show that L and $\langle\cdot, v\rangle$ agree on $\ker L$ and $\operatorname{span}(v)$.

On $\ker L$ they obviously agree. On $\operatorname{span}(v)$, it's by construction that they agree.

Problem 6

Suppose $f \in L^2$ and the Fourier transform is non-negative for almost every ξ. Show that the set of finite linear combinations of translates of f is dense in L^2.

Proof

Suppose not. Then there is $0 \neq g \in L^2$ with $\langle f_a, g\rangle = 0$ for all $f_a = \tau_a f$ translates. By Plancherel,

$$\begin{aligned} 0 = \int \hat{f}_a \hat{g} &= \int e^{-2\pi i a\xi} \hat{f}\hat{g} \\ &= \widehat{\hat{f}\hat{g}}(a) \end{aligned}$$

and so $\hat{f}\hat{g}$ must be identically zero, so $\hat{g} = 0$ and so $g = 0$, a contradiction.

Problem 7

Let $\{u_n\}$ be a sequnece of real-valued harmonic functions on $\mathbb{D}$ that obey

$$u_1 \geq u_2 \geq \cdots \geq 0$$

on $\mathbb{D}$. Prove that $\inf u_n$ is harmonic on $\mathbb{D}$.

Proof

First we prove local uniform convergence by Harnack's inequality. Fix $r < 1$. For $z \in B(0, r)$, we have that $u_n - u_m$ is positive and harmonic, and so for $r < R < 1$,

$$(u_n - u_m)(z) \leq \frac{R + r}{R - r}(u_n - u_m)(0)$$

which goes to 0 uniformly as $n, m \to \infty$.

Now that we have locally uniform convergence, we apply the mean value property and swap the limit and integral with uniform convergence:

$$⨍ u(z_0 + re^{i\theta})\, d\theta = \lim ⨍ u_n(z_0 + re^{i\theta})\, d\theta = \lim u_n(z_0) = u(z_0)$$

so it obeys the mean value property, and is therefore harmonic.

Alternatively, dominated convergence gives the integral claim immediately. This is enough, since L^1_{loc} functions satisfying the mean value property are harmonic.

Problem 8

Let Ω be the following subset of the complex plane:

$$\Omega := \{x + iy \mid x > 0, y > 0, xy < 1\}$$

Give an example of an unbounded harmonic function on Ω that extends continuously to the boundary and vanishes there.

Proof

Note that $z \mapsto \pi z^2$ sends this domain to the horizontal strip $\mathbb{R} \oplus i[0, 2\pi]$. So just take $\operatorname{Im} \exp(\pi z^2)$ and we're done.

Problem 9

Prove Jordan's lemma: If $f : \mathbb{C} \to \mathbb{C}$ is meromorphic, $R > 0$ and $k > 0$ then

$$\left|\int_\Gamma f(z)e^{ikz}\,dz\right| \le \frac{100}{k}\sup_{z\in\Gamma}|f(z)|$$

where Γ is the quarter circle $z = Re^{i\theta}$ where $0 \le \theta \le \pi/2$. (It is possible to replace 100 here by $\pi/2$ but you are not required to prove that.)

Proof

Let's just start.

$$\begin{aligned}\left|\int_\Gamma f(z)e^{ikz}\,dz\right| &\le \sup_\Gamma|f(z)|\int\left|e^{ikz}\right|dz\\ &= \sup_\Gamma|f(z)|R\int_0^{\pi/2} e^{-kR\sin(\theta)}\,d\theta\end{aligned}$$

Note that $\sin(\theta) \ge 2/\pi$ on $0 < \theta < \pi/2$, so we have

$$\begin{aligned}\left|\int_\Gamma f(z)e^{ikz}\,dz\right| &\le \sup_\Gamma|f(z)|R\int_0^{\pi/2} e^{-kR\theta\cdot(2/\pi)}\,d\theta\\ &= \sup_\Gamma|f(z)|\frac{\pi}{2k}\left(1-e^{-kR}\right)\\ &\le \frac{\pi}{2k}\sup_\Gamma|f(z)|\end{aligned}$$

Problem 10

Let us define the Gamma function via

$$\Gamma(z) = \int_0^\infty t^z e^{-t}\frac{dt}{t}$$

at least when the integral is absolutely convergent. Show that this function extends to a meromorphic function.

Proof

First, let's show that it's finite for $\mathrm{Re}(z) > 0$. We note that

$$\begin{aligned} |\Gamma(z)| &\leq \int_0^\infty \left|t^{z-1}\right| e^{-t}\, dt \\ &\leq \int_0^\infty t^{\mathrm{Re}\, z-1} e^{-t}\, dt \end{aligned}$$

This integral can be shown to converge by considering $0 < t < 1$ on which $e^{-t} \leq 1$ and considering $1 < t < \infty$ on which $t^{\mathrm{Re}\, z-1} < e^{-t/2}$ and so we get convergence.

Next, we need to show that this is holomorphic on this half-plane. For $\mathrm{Re}(z) > 1$ and $|h| < 1$, we have

$$\frac{1}{h}\left(\Gamma(z+h) - \Gamma(z)\right) = \int_0^\infty \frac{t^{z+h-1} - t^{z-1}}{h} e^{-t}\, dt$$

Taking $h \to 0$, we observe that the integrand converges to $(z-1)t^{z-2}$ which is integrable against e^{-t}. But it's not enough that it converges to something good, it needs to be eventually good.

Let's write

$$\frac{t^{z+h-1} - t^{z-1}}{h} = t^{z-1}\left|\frac{t^h - 1}{h}\right|$$

When $|h| \leq 1$, we'll use the series expansion to understand the integrand

$$\begin{aligned} \left|\frac{1}{h}(t^h - 1)\right| = \left|\frac{1}{h}(e^{h \log t} - 1)\right| &= \sum_{n=0}^\infty |h|^{n-1} \left|\log t\right|^n / n! \\ &\leq e^{\left|\log t\right|} \end{aligned}$$

Since $t^{z-1} e^{\left|\log t\right| - t}$ is integrable, we can apply the Lebesgue dominated convergence theorem to show that Γ is differentiable on $\mathrm{Re}\, z > 1$.

Note that $\Gamma(1) = 1$.

Next, we'll prove an identity: $\Gamma(z+1) = z\Gamma(z)$. This follows by integration by parts for $\mathrm{Re}\, z > 0$

$$\begin{aligned} \Gamma(z+1) &= \int t^z e^{-t}\, dt \\ &= t^z e^{-t}\Big|_0^\infty + \int_0^\infty z t^{z-1} e^{-t}\, dt \\ &= z\Gamma(z) \end{aligned}$$

Let's use this to extend strip-wise. Set $\Gamma(z) = \frac{1}{z}\Gamma(z+1) = \frac{1}{z(z+1)}\Gamma(z+2)$ for all $-1 < \operatorname{Re} z \le 0$ except for $z = 0$. This is analytic except at 0.

We extend like this to all of $\mathbb{C}$. Because the two functions overlap, there are no issues on $\operatorname{Re} z \in \mathbb{Z}$. It's thus analytic everywhere except the non-positive integers.

To show it's meromorphic, we just see that $\Gamma(z) = \frac{1}{z(z+1)\ldots(z+n+1)}\Gamma(z+n+2)$ and observe that $\Gamma(1) = 1$, so $\Gamma(-n)$ must be a pole of order n.

Problem 11

Let $P(z)$ be a polynomial. Show that there is an integer n and a second polynomial $Q(z)$ so that

$$P(z)Q(z) = z^n |P(z)|^2$$

on $|z| = 1$.

Proof

It suffices to provide a polynomial Q such that $Q(z) = z^n \overline{P(z)}$ on $|z| = 1$.

Let $P(z) = (z - a_1)(z - a_2)\ldots(z - a_n)$. Then

$$z^n \overline{P(z)} = (|z|^2 - \overline{a_1} z)(|z|^2 - \overline{a_2} z)\ldots(|z|^2 - \overline{a_n} z)$$

so take $Q(z) = (1 - \overline{a_1} z)(1 - \overline{a_2} z)\ldots(1 - \overline{a_n} z)$.

Problem 12

Show that the only entire function $f(z)$ obeying both

$$|f'(z)| \le e^{|z|} \text{ and } f\left(\frac{n}{\sqrt{1+|n|}}\right) = 0$$

for all $n \in \mathbb{Z}$ is the zero function.

Proof

Note that f' has order at most 1, so $\sum 1/|a_n|^2 < \infty$ where the zeroes are a_n. Write $g'(z) = f'(z)\overline{f'(\bar{z})}$ which is now real on the real axis. This has the same order, and so it has zeroes on intervals that go like $n/\sqrt{1+|n|}$. In particular, $\sum 1/|a_n|^2 \sim \sum 1/n = \infty$ diverges. This is a contradiction, so f' must be everywhere zero, so f is constant.

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Problem 1

Let $1 < p < \infty$ and $f_n : \mathbb{R}^3 \to \mathbb{R}$ be st $\limsup \|f_n\|_{L^p} < \infty$. Suppose f_n converge ae pointwise. Show they converge weakly in L^p.

Proof

Banach-Alaoglu says a bounded set is weakly-* compact, but L^p is reflexive, (and weakly compact is also weakly sequentially compact in this case) so there is a weakly convergent subsequence.

But every subsequences has a further subsequence which converges to f, so in fact $f_n \rightharpoonup f$.

Problem 2

Suppose μ is a Borel probability measure on the unit circle in the complex plane such that

$$\lim_{n\to\infty} \int_{|z|=1} z^n \, d\mu(z) = 0$$

For $f \in L^1(\mu)$, show that

$$\lim_{n\to\infty} \int_{|z|=1} z^n f(z) \, d\mu(z) = 0$$

Proof

First, it's clear by linearity that the result holds for trigonometric polynomials. Note that trigonometric polynomials are dense in continuous functions (uniformly) which are dense in L^1.

Next, take $\|f - g\|_{L^1} < \varepsilon$ where $g \in C(\mathbb{T})$ and $\|P - g\|_{L^\infty} < \varepsilon$ where P is a trigonometric polynomial.

Take N such that $\left|\int_{\mathbb{T}} z^n P(z) \, d\mu(z)\right| < \varepsilon$ for all $n \geq N$. Then

$$\begin{aligned}\left|\int_{\mathbb{T}} z^n f(z) \, d\mu\right| &\leq \left|\int_{\mathbb{T}} z^n (f(z) - g(z)) \, d\mu\right| + \left|\int_{\mathbb{T}} z^n (g(z) - P(z)) \, d\mu\right| + \varepsilon \\ &\leq \|f - g\|_{L^1} + \|g - P\|_{L^1} + \varepsilon < 3\varepsilon\end{aligned}$$

Problem 3

Let H be a Hilbert space and E a closed convex subset of H. Prove that there exists a unique element $x \in E$ such that $\|x\| = \inf_{y \in E} \|y\|$.

Proof

This is a classic problem.

First, we show uniqueness. If x, y both satisfy the desired property, then consider $(x+y)/2$ which has a smaller norm by the parallelogram law and is also a member of E.

Take an infimizing sequence $y_n \in E$ with $\|y_n\| \to \inf_E \|y\| = m$. Note that $(y_n + y_m)/2 \in E$ so the parallelogram law gives us

$$\begin{aligned} \frac{1}{2}\|y_n - y_m\|^2 &= \|y_m\|^2 + \|y_m\|^2 - \frac{1}{2}\|y_n + y_m\|^2 \\ &\leq 2m^2 - 2\|(y_n + y_m)/2\|^2 = 0 \text{ in the limit} \end{aligned}$$

so in fact $y_n \to y$ which has the desired properties.

Problem 4

Fix $f \in C(\mathbb{T})$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Let s_n denote the n-th partial sum of the Fourier series of f. Prove that

$$\lim_{n \to \infty} \frac{\|s_n\|_{L^\infty(\mathbb{T})}}{\log n} = 0$$

Proof

First we'll show that $\|s_n(f)\| \lesssim \|f\|_{L^\infty} \log n$ by considering the Dirichlet kernel. Next, we'll take P close to f in L^∞ where $\|s_n(P)\|/\log n \to 0$.

Recall that $s_n(f) = f * D_n$ where

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt} = \frac{\sin((n+1/2)t)}{\sin(t/2)}$$

We note that $\|s_n(f)\|_{L^\infty} \le \|f\|_{L^\infty}\|D_n\|_{L^1}$ immediately, so we estimate

$$\begin{aligned}
\|D_n\|_{L^1} &\lesssim \int \left|\frac{\sin((n+1/2)t)}{\sin(t/2)}\right| \lesssim \int \left|\frac{\sin((n+1/2)t)}{t}\right| \\
&\lesssim \int_0^{(n+1/2)\pi} \frac{|\sin u|}{u}\,du \lesssim \sum_{k=0}^{n} \int_{k\pi}^{(k+1)\pi} \frac{|\sin u|}{u}\,du \\
&\lesssim \sum_{k=0}^{n} \frac{1}{k+1} \lesssim \log n
\end{aligned}$$

Finally, note that if P is a trig polynomial, then $s_n P \to P$ uniformly. By taking P such that $\|f - P\|_{L^\infty} < \varepsilon$ we're done.

Problem 5

Let $f_n : \mathbb{R}^3 \to \mathbb{R}$ be a sequence of functions such that $\sup_n \|f_n\|_{L^2} < \infty$. Show that if $f_n \to f$ ae, then

$$\int_{\mathbb{R}^3} \left||f_n|^2 - |f_n - f|^2 - |f|^2\right| dx \to 0$$

Proof

First of all, note that $\|f\|_{L^2} \le \sup_n \|f_n\|_{L^2} =: M$ by Fatou's lemma. We next rewrite

$$\begin{aligned}
\left||f_n|^2 - |f_n - f|^2 - |f|^2\right| &= \left||f_n + f - f|^2 - |f_n - f|^2 - |f|^2\right| \\
&= 2|f_n - f||f|
\end{aligned}$$

Let's split up the integral with Egorov and note that the integral of $|f|^2$ on small sets is small.

$$\begin{aligned}
\int |f_n - f||f| &\le \int_{B(0,R)} |f_n - f||f| + \int_{B(0,R)^c} |f_n - f||f| \\
&\le \int_{B(0,R)\cap E} |f_n - f||f| + \int_{B(0,R)\setminus E} |f_n - f||f| + 2M\varepsilon \\
&\le M\|f_n - f\|_{L^\infty} + 2M\varepsilon + 2M\varepsilon
\end{aligned}$$

as desired.

Problem 6

Let $f \in L^1(\mathbb{R})$ and let Mf denote its maximal function, i.e.,

$$(Mf)(x) = \sup_{0<r<\infty} \frac{1}{2r} \int_{-r}^{r} |f(x-y)|\, dy$$

By the Hardy-Littlewood maximal function theorem (the weak type L^1 bound), we have

$$|\{Mf(x) > \lambda\}| \lesssim \frac{\|f\|_{L^1}}{\lambda}$$

(the constant is 3) for all $\lambda > 0$.

Using this, show

$$\limsup_{r\to 0} \fint_{x-r}^{x+r} |f(y) - f(x)|\, dy = 0$$

for ae $x \in \mathbb{R}$.

Proof

(The limits of integration must be $x-r$ to $x+r$ or the question is false as stated.)

Let $\varepsilon > 0$. First, let $g \in C_c$ be such that $\|f-g\|_{L^1} < \varepsilon$. Write $h = f - g$. Then $f = g + h$ and since $T_r f := \fint_{x-r}^{x+r} |f(y) - f(x)|\, dy$ is a sublinear operator, we have

$$T_r f \le T_r g + T_r h$$

but $T_r g \to 0$ as $r \to \infty$ everywhere by continuity. Thus we need only consider $T_r h$. Define $Th = \limsup_r T_r h$. If $T_r h(x) > \lambda$, let's consider what this implies. We bound

$$\begin{aligned}
T_r h &= \frac{1}{2r} \int_{x-r}^{x+r} |h(y) - h(x)|\, dy \\
&\le \frac{1}{2r} \int_{x-r}^{x+r} |h(y)| + |h(x)| \\
&= M_r h(x) + |h(x)|
\end{aligned}$$

and so if $T_r h > \lambda$, then $M_r h > \lambda/2$ or $|h(x)| > \lambda/2$. Thus taking limsups we get

$$\{Thx > \lambda\} \subseteq \{Mh > \lambda/2\} \cup \{|h| > \lambda/2\}$$

which by Chebyshev and the Hardy-Littlewood maximal inequality gives us

$$|\{Th > \lambda\}| \leq \frac{\|h\|_{L^1}}{\lambda} \leq 2\varepsilon/\lambda$$

which is true for any ε, so $Th = 0$ almost everywhere as desired.

Problem 7

Let f be holomorphic in $\mathbb{C}$ and suppose that $f(0) = 0$, $f(1) = 1$, and $f(\mathbb{D}) \subseteq \mathbb{D}$. Show that

1. $f'(1) \in \mathbb{R}$
2. $f'(1) \geq 1$.

Proof, Part 1

Suppose $f'(1)$ isn't real. Then there is some $v \in \mathbb{C}$ with $\mathrm{Re}(v) < 0$ but $\mathrm{Re}(vf'(1)) > 0$. Then

$$\lim_{t \to 0} \frac{f(1+tv) - f(1)}{t} = \lim_{t \to 0} \frac{f(1+tv) - 1}{t} = f'(1)v$$

but since this has a real part greater than zero, eventually $f(1+tv)$ has a real part greater than 1, so $f(1+tv) \notin \mathbb{D}$, but $1 + tv \in \mathbb{D}$.

Proof, Part 2

We consider the difference quotient

$$\begin{aligned} f'(1) &= \lim_{t \to 0} \frac{f(1-t) - f(1)}{t} \\ &= \frac{f(1-t) - 1}{t} \end{aligned}$$

and recall that $|f(1-t)| \leq |1-t|$ by the Schwarz Lemma and so $|f'(1)| \geq 1$.

Problem 8

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function such that every zero of f has even multiplicity.

Show that f has a holomorphic square root, i.e., there exists $g : \mathbb{C} \to \mathbb{C}$ such that $f = g^2$.

Proof

This is a consequence of the monodromy theorem.

Without loss of generality, let $f(0) = a \neq 0$ and let α be some square root of a.

In some neighborhood of 0, there is a unique choice of g such that $g^2 = f$ and $g(0) = \alpha$.

In general, on every sufficiently small open set away from any zeroes of f, there are two choices for g.

I claim that there is a unique analytic extension to $\mathbb{C}$ of the function g we defined near the origin. We can clearly extend g along any path that doesn't intersect any zeroes of g, but for it to be a single-valued function, we need only verify that loops around the zeroes of f don't change the value.

This is by the monodromy theorem, which says that the extension depends only on the homotopy class of the curve along which we extend.

So let w be a zero of f. Since w is an even-ordered zero, $f(z) = (z - w)^{2k}h(z)$ where h is non-zero near w. Thus h has a well-defined square root in a neighborhood of w, which we'll call ϕ, and so we must select $g(z) = \pm(z-w)^k\phi(z)$ near w.

We can now extend g analytically along any curve in the plane at all, even if it intersects the zeroes of f, so since $\pi_1(C) = 0$, there is a single-valued choice of extension.

Alternate Proof

We can construct a function h by the Weierstrass theorem on canonical products with the same zeroes as f but only half the multiplicity. Thus f/h^2 is analytic everywhere with no zeroes, so it has a logarithm $\log(f/h^2) = g$.

But then $f = h^2 \exp(g)$, so $h \exp(g/2)$ is a square root.

Problem 9

Suppose f is holomorphic in $\mathbb{D}$ and $0 < x_{n+1} < x_n < 1$ is a sequence of real numbers with $x_n \to 0$. Show that if $f(x_{2n+1}) = f(x_{2n})$ for all n, then f is constant.

Proof

Without loss of generality, $f(0) = 0$ (by translating the function vertically).

Write $g(z) = f(z)\overline{f(\overline{z})}$. This is real on $\mathbb{R}$ and satisfies the same properties and is holomorphic. Now $g'(z) = 0$ by the mean value theorem at points between

x_{2n+1} and x_{2n}, so $g' = 0$ everywhere because it's true on a set with a limit point. Thus $g = 0$ everywhere.

By the pigeonhole principle, either $f = 0$ on a set with a limit point, or $\overline{f(\bar{z})}$ is. Thus $f = 0$ identically, as desired.

Problem 10

Let $\{f_n\}$ be a sequence of holomorphic functions on $\mathbb{D}$ satisfying $|f_n(z)| \le 1$ for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Let $A \subseteq \mathbb{D}$ be the set of all $z \in \mathbb{D}$ for which the limit $\lim_n f_n(z)$ exists.

Show that if A has an accumulation point in $\mathbb{D}$, then there exists a holomorphic function f on $\mathbb{D}$ such that $f_n \to f$ locally uniformly on $\mathbb{D}$.

Proof

Note that $\{f_n\}$ is a normal family, so there is a subsequence f_{n_k} which converges locally uniformly to some function g. If A has an accumulation point, then $g(z) = \lim f_{n_k}(z)$ at all $z \in A$.

But note that for any subsequence of f_n, there is a further subsequence which converges to some g_α that agrees with g on A, so $g_\alpha = g$. Thus every subsequence has a further subsequence converging locally uniformly to g and therefore $f_n \to g$ locally uniformly.

Problem 11

Find all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ satisfying $f(z+1) = f(z)$ and $f(z+i) = e^{2\pi} f(z)$.

Proof

Consider $g(z) = e^{-2\pi i z}$. Then g has the same periodicity conditions. Furthermore, g is never zero, so f/g is analytic.

Note that $(f/g)(z+1) = (f/g)(z)$ and $(f/g)(z+i) = e^{2\pi - 2\pi}(f/g)(z)$, so f/g is doubly periodic and holomorphic and thus constant. Thus $f = cg$ for some constant c.

Problem 12

Let $M \in \mathbb{R}$, $\Omega \subseteq \mathbb{C}$ be a bounded open set, and $u : \Omega \to \mathbb{R}$ be harmonic.

1. Show that if

$$\limsup_{z \to z_0} u(z) \leq M$$

for all $z_0 \in \partial\Omega$, then $u(z) \leq M$ for all $z \in \Omega$.

2. Show that if u is bounded from above and there exists a finite set $F \subseteq \partial\Omega$ such that the above condition is valid for all $z_0 \in \partial\Omega \backslash F$ then the conclusion of the first part still holds.

Proof, Part 1

This is an application of the maximum principle.

Suppose $u(z) = M + \varepsilon$. For each point $z_0 \in \partial\Omega$, take an open neighborhood on which u is at most $M + \varepsilon/2$. This open neighborhoods cover $\partial\Omega$ and hence we can take finitely many of them, $U_1, \ldots U_n$. The complement of the union of their closure in Ω, i.e., $\Omega \setminus \bigcup \overline{U_i}$ is an open set with u on the boundary at most $M + \varepsilon/2$ but u in the interior attains a maximum at least $M + \varepsilon$. This contradicts the maximum principle.

Proof, Part 2

Let's force Part 1 to hold by modifying our function. Let $p_1, \ldots, p_n$ be the points on the boundary in question. We want to sent $u(p_i)$ to $-\infty$ by modifying u.

Let d be the diameter of Ω. Define $v(z) = -\sum_i \log((z - p_i)/d)$. This is harmonic and non-negative.

Now consider $f_\varepsilon(z) = u(z) - \varepsilon v(z)$. This satisfies Part 1 at every point on the boundary for every ε, so sending $\varepsilon \to 0$ gives us $u(z) \leq M$ everywhere.

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Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ be bounded, Lebesgue measurable, and

$$\lim_{h \to 0} \int_0^1 \frac{|f(x+h) - f(x)|}{h}\, dx = 0$$

Show that f is a.e. constant on $[0, 1]$.

Proof

We can't swap limits. This looks like a Lebesgue differentation theorem, but need to change functions.

Try defining $F(x) = \int_0^x f$. Note that $F \in L^\infty \subseteq L^1$, so we can apply the Lebesgue differentiation theorem, which says that

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

for almost every x. Pick good a and b and mess around with $|f(a) - f(b)|$. Eventually get it's equal to zero.

Let A be the set of points x where that limit does converge as desired. Pick $a, b \in A$. Then

$$\begin{aligned}
|f(a) - f(b)| &= \left|\lim_{h\to 0} \frac{F(a+h) - F(a)}{h} - \lim_{h\to 0} \frac{F(b+h) - F(b)}{h}\right| \\
&= \lim_{h\to 0} \frac{1}{h}\left|\int_a^b f(x+h)\,dx - \int_a^b f(x)\,dx\right| \\
&\le \lim_{h\to 0} \frac{1}{h} \int_0^1 |f(x+h) - f(x)| \to 0
\end{aligned}$$

so $f(a) = f(b)$.

Problem 2

Consider $\ell^2(\mathbb{Z})$. Show that the Borel σ-algebra $\mathcal{N}$ associated to the norm topology agrees with the Borel σ-algebra $\mathcal{W}$ associated to the weak topology.

Proof

Note that the weak topology is coarser than the norm topology, since the weak topology is the coarsest such that $x \mapsto \langle h, x\rangle$ is continuous for any $h \in \ell^2$. These are all continuous in norm topology by Cauchy-Schwarz.

Let's pick an orthonormal basis $\{e_n\}$. Write $\|v\|^2 = \sum |\langle e_n, v\rangle|^2$ as the sum of $\mathcal{W}$-measurable functions.

We show that the norm is $\mathcal{W}$-measurable, and so the pre-image of the ball of this function is $\mathcal{W}$-measurable.

Thus $\mathcal{W}$ contains any open ball. This tells us $\mathcal{N} = \mathcal{W}$.

In general, separability of X^* should be enough.

Problem 3

Given $f : \mathbb{R}^2 \to \mathbb{R}$ continuous, define

$$\begin{aligned} A_r f(x,y) &= \fint_{-\pi}^{\pi} f(x + r\cos\theta, y + r\sin\theta)\, d\theta \\ Mf(x,y) &= \sup_{0<r<1} A_r f(x,y) \end{aligned}$$

By a theorem of Bourgain, there is an absolute constant C such that

$$\|Mf\|_{L^3(\mathbb{R}^2)} \le C\|f\|_{L^3(\mathbb{R}^2)} \quad \forall f \in C_c(\mathbb{R}^2)$$

Use this to show that if K is compact, then $A_r\chi_K \to 1$ as $r \to 0$ pointwise almost everywhere in K.

Proof

Let $g \in C_c$, $f = \chi_K$ and $\|g - f\| < \varepsilon$. Then

$$|A_r(f) - f(x)| \le |A_r(f - g)| + |A_r(g) - g(x)| + |f(x) - g(x)|$$

because A_r is linear in f.

Write $|A_r(f - g)| \le M(|f - g|)$. Recall Chebyshev's inequality, that

$$m(f > \lambda) \le \|f\|_{L^p}^p / \lambda^p$$

(Proof is trivial. Integrate over the set $f > \lambda$.)

Then $m(M(|f - g|) > \lambda) \le C^3\varepsilon^3/\lambda^3$ Also $|f - g| > \lambda$ outside ε^3/λ^3.

Thus $|A_r(f) - f(x)| > 2\lambda$ outside a set of measure $(C^3 + 1)\varepsilon^3/\lambda^3$. This assumes A_r is defined for L^3 functions in a way that extends the inequality.

It's enough to show it for functions that look like characteristic functions of compact sets, which are probably nicer than general L^3 functions. There is no solution given for this.

Send $\varepsilon \to 0$. Then we get $\lim_{r\to 0}|A_r(f) - f(x)| \le 2\lambda$ outside a set of measure 0, i.e., almost everywhere. Sending $\lambda \to 0$ gives us the result.

Basic strategy is apply Chebyshev.

Other Idea

Let $g \in C_c$, $f = \chi_K$ and $\|g - f\|_{L^3} < \varepsilon$. Then

$$|A_r(f)(x) - f(x)| \le |A_r f(x) - A_r g(x)| + |A_r g(x) - g(x)| + |f(x) - g(x)|.$$

Taking cubes and integrals gives us

$$\begin{aligned}\|A_r f(x) - f(x)\|_{L^3} &\le \|A_r f - A_r g\|_{L^3} + \|A_r g - g\| + \|f - g\| \\ &\le C\varepsilon + \varepsilon + \|A_r g - g\|_{L^3}\end{aligned}$$

Note that $\|A_r g - g\|_{L^3} \le \|A_r g - g\|_{L^\infty}$, which goes to zero by uniform continuity.

This still requires the inequality for more general f, though (in particular, differences of continuous functions and characteristic functions).

Problem 4

Let K be a non-empty compact subset of $\mathbb{R}^3$. For any Borel probability measure μ on K, define the Newtonian energy $I(\mu) \in (0, \infty]$ by

$$I(\mu) = \int_K \int_K \frac{1}{|x - y|} \, d\mu(x) \, d\mu(y)$$

and let R_K be the infimum of $I(\mu)$ overall Borel probability measures μ on K.

Show that there exists μ such that $I(\mu) = R_K$.

Proof

M is the set of all Borel probability measures on K. This is the unit ball in $C(K)^*$. Let μ_n be a sequence with $I(\mu_n) \to R_K$. Since $C(K)$ is separable, the unit ball in the dual is weakly-star (sequentially!) compact. Thus after passing to a subsequence, there's μ with $\mu_n \overset{*}{\rightharpoonup} \mu$.

Applying weak-star convergence to the constant function 1, μ is also a probability measure. Let's show that $I(\mu) = R_K$.

First, need to show that $\mu_n \otimes \mu_n \overset{*}{\rightharpoonup} \mu \otimes \mu$ in weak-star of $C(K \times K)$. Clearly $\iint f \, d\mu_n(x) \, d\mu_n(y) \to \iint f \, d\mu(x) \, d\mu(y)$ if $f(x, y) = g(x)h(y)$. But the span of these functions is obviously dense by Stone-Weierstrass (write any f like a sum of a product of two monomials, one in x and one in y).

But $\frac{1}{|x-y|}$ isn't continuous. It is USC though, so we get $\liminf I(\mu_n) \ge I(\mu)$ by the *portmanteau theorem*. This is enough. This can be shown directly without much trouble by playing with limits.

Problem 5

Define $H = \{u : \mathbb{D} \to \mathbb{R} \mid u \text{ is harmonic and } \int_{\mathbb{D}} |u|^2 < \infty\}$. Show that $f \mapsto \partial_x f(0, 0)$ is a bounded linear functional and compute its norm.

Proof

On the disc, harmonic means real part of holomorphic, since simply connected.

Write $u(re^{i\theta}) = \sum \mathrm{Re}(a_n r^n e^{in\theta}) = \sum r^n \mathrm{Re}(a_n)\cos(n\theta) - \mathrm{Im}(a_n)\sin(n\theta)$.

Note that $\partial_x u = \mathrm{Re}(f')$, so $u_x(0) = \mathrm{Re}(a_1)$. So how big can $\mathrm{Re}(a_1)$ get?

First of all, let's write

$$\begin{aligned}
\int_{\mathbb{D}} u^2 &= \int_0^1 \int_0^{2\pi} \left(\sum r^n \mathrm{Re}(a_n)\cos(n\theta) - \mathrm{Im}(a_n)\sin(n\theta)\right)^2 r\,d\theta\,dr \\
&= \int_0^1 r \int_0^{2\pi} \sum_{m,n} r^m r^n \dots \,d\theta\,dr \\
&= \int_0^1 \sum_n r^{2n+1}\pi|a_n|^2\,dr \\
&\geq \int_0^1 r^3 \mathrm{Re}(a_1)^2 = \pi\,\mathrm{Re}(a_1)^2/4
\end{aligned}$$

we use orthonormality of sin and cos. Thus $\mathrm{Re}(a_1) \leq \frac{2}{\sqrt{\pi}}\|u\|$ by taking square roots.

Try $f(z) = z$, so $u(x,y) = x$ and we achieve this bound, since $a_1 = 1$. This is obviously linear and now clearly continuous.

Problem 6

Let $X = \{\xi \mapsto \int_{\mathbb{R}} e^{i\xi x} f(x)\,dx \mid f \in L^1(\mathbb{R})\}$. This is the image of L^1 under the (inverse) Fourier transform. Show that X is a subset of C_0, is dense, and is not C_0.

Proof

To show continuity, immediately from dominated convergence. To show C_0, show for simple functions by calculating the integral for $(0,1)$ and thus for all open intervals. Then density and $\left\|\hat{f} - \hat{s}\right\|_{L^\infty} \leq \|f - s\|_{L^1}$ gives the result.

For density, note that it contains all Schwartz functions which are dense in L^1 and in C_0.

For non-surjectivity, since the Fourier transform is an injective bounded linear map from L^1 to C_0, surjectivity would imply the inverse map $C_0 \to L^1$ would be bounded.

But let's take $h = \chi_{[-1,1]}$ and $h_n \in C_c^\infty \to h$ in L^2. Let $g_n = \mathcal{F}^{-1}(h_n)$. These are Schwartz and hence in L^1. Thus $g_n \to g$ in L^2. Can assume $g_n \to g$ pointwise a.e. also.

Note that g can't be L^1, so $\|g_n\|_{L^1} \to \infty$, but this contradicts boundedness of $\mathcal{F}^{-1}$.

Alternatively for density, enough to show that if a linear functional on C_0 dies on $\mathcal{F}(L^1)$, then it is zero. Such a functional is given by integration against some μ a regular Borel measure. Then $\int \int e^{i\xi x} f(x)\, dx\, d\mu(\xi) = \iint e^{i\xi x} f(x)\, d\mu(\xi)\, dx$ by Fubini, so this is zero for each f. Thus $\hat{\mu} = 0$.

Nxt, try to show that $\mu = 0$. This is done by first showing that μ hitting any continuous periodic function because we can write them all as sums of exponentials. Now for $g \in C_c$ and g_L periodic agreeing with g on $[-L, L]$ (with period $2L$), write

$$\begin{aligned} \int g\, d\mu &= \int g_L\, d\mu + \int (g - g_L)\, d\mu \\ &\leq \varepsilon + \int_{|\xi|>L} (g - g_L)\, d\mu \\ &\leq \varepsilon + o(1) \text{ as } L \to \infty \end{aligned}$$

Problem 7

Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire (holomorphic) such that $\log|f|$ is L^1. Show that f is constant.

Proof

Suppose f is not constant. Then, since $\log|f(z)|$ is subharmonic, it satisfies the mean value property.

Suppose $\log|f(z_0)| > 1$. If this never happens, f is bounded and hence constant. Otherwise,

$$\begin{aligned} \int_{\mathbb{R}^2} \log|f(z)| &= \int_0^\infty r \int_0^{2\pi} \log|f(z_0 + re^{i\theta})|\, d\theta\, dr \\ &\geq \int_0^\infty 2\pi r \cdot 1\, dr = \infty \end{aligned}$$

because that θ integral is the average.

This only works if the average is at least 1, which only occurs by Liouville when f is non-constant.

Problem 8

Let A, B be positive definite $n \times n$ real symmetric marices with the property $\left\|BA^{-1}x\right\| \leq \|x\|$ for all $x \in \mathbb{R}^n$, where $\|\cdot\| = \|\cdot\|_{\ell^2([n])}$.

1. Show that for each pair $x, y \in \mathbb{R}^n$, $z \mapsto \langle y, B^z A^{-z} x\rangle$ admits an analytic continuation from $0 < z < 1$ to the whole complex plane.

2. Show that $\left\|B^\theta A^{-\theta} x\right\| \leq \|x\|$ for all $0 \leq \theta \leq 1$.

Proof, Part 1

Diagonalize and just write $A^{-z} = SD^{-z}S^{-1}$ and $B = RE^zR^{-1}$. This is a polynomial in the z-th powers of eigenvalues of A and B and inverses. Note that the eigenvalues are real and positive, so this is all good.

Proof, Part 2

Goal: Hadamard's three lines lemma.

Let $f_{x,y}(z) = z \mapsto \langle y, B^z A^{-z} x\rangle$. We have that on $\mathrm{Re}(z) = 0$,

$$\left|f_{x,y}(z)\right| \leq \|x\|\|y\|$$

since the eigenvalues of A^z and B^z have norm 1 (since anything to a purely imaginary power is unit length).

If $\mathrm{Re}(z) = 1$, then for $z = 1 + bi$,

$$\|B^z A^{-z}\| = \left\|B^{bi} B A^{-1} A^{-ib}\right\| \leq \|B^{bi}\|\left\|A^{-iz}\right\|\left\|BA^{-1}\right\| \leq 1$$

so $\left|f_{x,y}(z)\right| \leq \|y\|\|x\|$.

If $f_{x,y}$ is bounded, then the Hadamard three lines theorem tells us that $\left|f_{x,y}(z)\right| \leq \|y\|\|x\|$ on the strip. But f is bounded on the strip because each λ^z is bounded on the strip.

Problem 9

Let P be a non-constant polynomial, all of whose zeroes lie in a half plane $\{\mathrm{Re} > \sigma\}$. Show that all zeroes of P' also lie in the same half plane.

Proof

This is a standard exercise found in Ahlfors.

Write $P(z) = \prod(z - a_i)$. Then

$$\frac{P'}{P} = \frac{1}{z - a_1} + \cdots + \frac{1}{z - a_n}$$

If $P'(z) = 0$ then if $P(z) = 0$ then it's obviously true. Otherwise, take the real part of the above expression.

$$\begin{aligned} 0 &= \operatorname{Re}\left(\frac{1}{z - a_1}\right) + \dots \\ &= \frac{\operatorname{Re}(z) - \operatorname{Re}(a_1)}{|z - a_1|^2} + \dots \end{aligned}$$

Then

$$\operatorname{Re}(z) \sum \frac{1}{\left|z - a_j\right|^2} = \sum \frac{\operatorname{Re}(a_j)}{\left|z - a_j\right|^2} < \sigma \sum \frac{1}{\left|z - a_j\right|^2}$$

and we're done.

Problem 10

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Without using either of the Picard theorems, show that there exist arbitraily large complex numbers z for which $f(z)$ is a positive real.

Proof

Suppose otherwise. Fix a closed ball B_r centered at zero such that f doesn't hit the positive reals outside of B_r. By compactness, $|f|$ attains a maximum value R on B_r. Thus $f - R$ is holomorphic and avoids the positive real axis everywhere.

Then just conformally map the target region to the disk. It's holomorphic and bounded, hence constant by Liouville.

The conformal map may be $\phi(z) = \frac{\sqrt{z}-i}{\sqrt{z}+i}$.

Problem 11

Let $f(z) = -\pi z \cot(\pi z)$ be meromorphic on $\mathbb{C}$.

1. Locate all poles of f and determine their residues.

2. Show that for each $n \geq 1$, the coefficient of z^{2n} in the Taylor expansion of f about 0 coincides with

$$a_n = \sum_{k=1}^{\infty} \frac{2}{k^{2n}}$$

Proof, Part 1

Write cot as cos/sin. Then $-\pi z \cot(\pi z)$ has a simple pole at every non-zero integer, since those are the zeroes of $\sin(\pi z)$. There's no pole at 0.

To calculate the residue at n, we have

$$\begin{aligned}
\operatorname{Res}(f, n) &= \lim_{z \to n} \frac{-\pi z(z-n)\cos(\pi z)}{\sin(\pi z)} \\
&= \lim_{z \to n} -z\cos(\pi z)\frac{\pi(z-n)}{\sin(\pi(z-n))} \\
&= -n(-1)^n = (-1)^{n+1} n
\end{aligned}$$

Proof, Part 2

The other standard representation:

$$\begin{aligned}
\pi \cot(\pi z) &= \sum_{k=-\infty}^{\infty} \frac{1}{z-k} \\
&= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2} \\
-\pi z \cot(\pi z) &= -1 - \sum_{k=1}^{\infty} \frac{2z^2}{z^2-k^2}
\end{aligned}$$

Then $f(z) = g(z^2)$ for $g(z) = -1 - \sum_{k \geq 1} \frac{2z}{z-k^2}$. Now we need the coefficient of z^n in the power series for g.

Writing $h(z) = \sum_{k \geq 1} \frac{1}{z-k^2}$, we get

$$\begin{aligned}
g^{(n)}(0) &= -2 \sum_{j=0}^{n} \binom{n}{j} z^{(j)}(0) h^{(n-j)}(0) \\
&= -2\binom{n}{1} h^{(n-1)}(0)
\end{aligned}$$

We calculate $h^{(j)}$ term-by-term

$$h^{(j)}(0) = \sum_{k\geq 1}(-1)^j j!\,(0-k^2)^{-j-1}$$

Thus

$$\begin{aligned} g^{(n)}(0) &= -2n\sum_{k\geq 1}(-1)^{(n-1)}(n-1)!\,\frac{1}{(-k^2)^n} \\ &= n!\sum_{k\geq 1}\frac{1}{k^{2n}} \end{aligned}$$

as desired.

Problem 12

Let $f : \mathbb{H} \to \mathbb{H}$ be holomorphic satisfying

$$\lim_{y\to\infty} yf(iy) = i$$

$$|f(z)| \leq \frac{1}{\operatorname{Im}(z)} \text{ for all } z \in \mathbb{H}$$

1. For $\varepsilon > 0$ write $g_\varepsilon = \frac{1}{\pi}\operatorname{Im} f(x+i\varepsilon)$. Show that

$$f(z+i\varepsilon) = \int_{\mathbb{R}} \frac{g_\varepsilon(x)}{x-z}\,dx$$

2. Show that there exists a Borel probability measure μ on $\mathbb{R}$ such that

$$f(z) = \int_{\mathbb{R}} \frac{1}{x-z}\,d\mu(x)$$

Proof, Part 1

If u is harmonic in $\mathbb{H}$, then $u(x+iy) = \int_{\mathbb{R}} P_y(x-t)u(t)\,dt$ where $P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$ is the Poisson kernel for upper half plane.

Let $f_\varepsilon(z) = f(z+i\varepsilon)$ and then $f_\varepsilon = u_\varepsilon + iv_\varepsilon$. Note that u_ε and v_ε are harmonic. They extend continuously (holomorphically!) to the real line.

We want to show

$$(u_\varepsilon + iv_\varepsilon)(z) = \int_{\mathbb{R}} \frac{1}{\pi}\frac{v_\varepsilon(x)}{x-z}\,dx$$

Break RHS into real and imaginary parts. The statement for v_ε is just the Poisson kernel. We need to show

$$u_\varepsilon(a+bi) = \int_{\mathbb{R}} \frac{v_\varepsilon(x)}{\pi} \frac{x-a}{(x-a)^2+b^2}\, dx$$

By applying the hypotheses to say $\lim_{b\to\infty}(b+\varepsilon)v_\varepsilon(ib) = 1$ and then by Fatou with the Poisson kernel to say that

$$1 = \lim_{b\to\infty}(b+\varepsilon)v_\varepsilon(ib) = \lim(b+\varepsilon)\int \frac{v_\varepsilon(x)b}{\pi(x^2+b^2)}\, dx \geq \int v_\varepsilon/\pi$$

Then dominated convergence lets just take a limit and dominate by $2v_\varepsilon$.

We use Cauchy-Riemann now. $\partial_a u_\varepsilon = \partial_b v_\varepsilon$ and $\partial_b u_\varepsilon = -\partial_a v_\varepsilon$. Expand v_ε using the formula we have and differentiate under the integral sign. Then integrate in a and b and compare, giving us

$$u_\varepsilon(a+bi) = \int_{\mathbb{R}} \frac{v_\varepsilon(x)}{\pi} \frac{(x-a)}{(x-a)^2+b^2}\, dx + C$$

for some real constant C.

Plugging in $\lim_{b\to\infty} u_\varepsilon(ib)(b+\varepsilon) \to 0$ means $C = 0$.

Proof, Part 2

Define $\mu_\varepsilon = g_\varepsilon\, dx$. We have $\frac{1}{x-z} \in C_0(\mathbb{R})$ for $\operatorname{Im}(z) > 0$. As μ_ε are probability measures (since v_ε/π is unit norm), Banach Alaoglu gives us a weakly-* convergent subsequence: $\mu_{\varepsilon_k} \overset{*}{\rightharpoonup} \mu$.

Then $f(z+i\varepsilon_k) = \left\langle \frac{1}{x-z}, \mu_{\varepsilon_k} \right\rangle \to \left\langle \frac{1}{x-z}, \mu \right\rangle = f(z)$. Just need to check that μ is a probability measure.

Note that $v_\varepsilon(bi) \to v(bi)$ as $\varepsilon \to 0$. Thus

$$v(bi) = \left\langle \frac{b}{x^2+b^2}, \mu \right\rangle$$

Thus our first condition tells us

$$1 = \lim_{b\to\infty} bv(bi) = \left\langle \frac{b^2}{x^2+b^2}, \mu \right\rangle \geq \langle 1, \mu \rangle$$

(where this last pairing isn't C^0, but tells us that $\mu(\mathbb{R}) \leq 1$. Thus $\mu(\mathbb{R}) = 1$ by Lebesgue dominated.

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Problem 1

Let $f : U \to \mathbb{C}$ be holomorphic and proper as $f : U \to V$. Show that f is surjective. Note U, V are connected.

Proof

By the open mapping theorem, $f(U)$ is open. To show f is closed as well, mess with limits and proper-ness.

Alternatively, try showing that $V \setminus f(U)$ is open. Take a point $x \in f(U)^c$ and look at a decreasing collection of closed balls around x.

Problem 2

Show that there is no function f holomorphic near $0 \in \mathbb{C}$ satisfying

$$f(1/n^2) = (n^2 - 1)/n^5$$

Proof

We use Taylor series. We get $f(0) = 0$, and learn that $f'(0) = 0$ as well.

So f/z^2 is holomorphic, but this is a contradiction to our hypothesis. This is a general style of question, so it's good to remember this kind of thing.

Alternatively, $f(x^2) = x^3 - x^5$ on a set with a limit point, so $f(z^2) = z^3 - z^5$, and so $f((-z)^2) = -z^3 + z^5$, a contradiction.

Problem 3

Does there exist $f : \mathbb{D} \to \mathbb{C}$ such that $|f(z_n)| \to \infty$ for all sequences approaching the boundary of the disk.

Proof

The goal is to take $1/f$ and argue something about the maximum principle. We need to clear out the poles of $1/f$ first (i.e., the zeroes of f) with a Blaschke product.

There are finitely many zeroes in the disc (otherwise they accumulate at the boundary, a contradiction).

So divide by the Blaschke product

$$B = \prod \frac{a_i - z}{1 - \bar{a}_i z}$$

and get f/B. But if we look at B/f, then there are no zeroes, and B/f goes to zero everywhere on the boundary and has no zeroes anywhere. This contradicts the maximum principle.

Alternatively, consider $\log|B/f|$ which is $-\infty$ on the boundary and finite on the inside. This also contradicts the maximum princple, but for harmonic functions.

Problem 4

Let u be non-negative and continuous on $\overline{\mathbb{D}} \setminus \{0\}$ which is harmonic on the interior. Let u be zero on the boundary and

$$\lim_{r \to 0} \frac{1}{r^2 \log r} \int_{0<|z|<r} u(z) = 0$$

Show that u is zero.

Proof

Let's show that $u/\log|1/z| \to 0$ as $z \to 0$.

Let $\varepsilon > 0$. Pick z so small that

$$\int_{0<|w|<3|z|/2} u < \varepsilon |z|^2 \log|1/z|$$

The mean-value property gives us

$$\begin{aligned} u(z) &\leq \frac{1}{\pi(|z|/2)^2} \int_{|w-z|<|z|/2} u(w) \leq \frac{4\pi}{|z|^2} \int_{0<|w|<3|z|/2} u(w) \\ &\leq \frac{4\pi\varepsilon|z|^2 \log|1/z|}{|z|^2} \end{aligned}$$

This implies that $u < \varepsilon \log|1/z|$ for sufficiently small z depending on ε.

We can get this another way. Let $v = \alpha \log(1/r)$ which is harmonic in the punctured disc. Thus $v \geq u$ everywhere by subharmonicity. Then

$$\begin{aligned} \int_{B(0,r)} u \leq \int_{B(0,r)} v &= 2\pi\alpha \left(\frac{r^2}{2} \log(1/r) + \frac{r^2}{4} \right) \\ \fint u &\leq \pi\alpha \log(1/r) + \frac{1}{2} \end{aligned}$$

Applying the mean value property gives us $u(0) \lesssim c\log|1/z|$ for sufficiently small z depending on choice of c, although this is a little false, since $u(0)$ isn't defined. This means $u = o(\log|1/z|)$ (once we mess around by picking $u(z)$ instead of $u(0)$).

Now let's consider $u - v$ which is still subharmonic, but goes to $-\infty$ at 0. But then at some small radius r, we need $u - v \le 0$ within $B(0, r)$, and the maximum principle says on the annulus this is true too. Thus $u - v \le 0$ for any choice of α. Thus $u \le 0$ by sending $\alpha \to 0$. But if $u \ge 0$ and $u \le 0$, then $u = 0$ identically, as desired.

Easier Proof

Averages on circles of harmonic functions go like $\alpha \log r + \beta$.

We need $\beta = 0$ by the boundary conditions. Next, we rewrite the integral and see

$$\frac{1}{r^2 \log r} \int_{0<|z|<r} u(z) = \cdots = \alpha\pi \left(1 - \frac{1}{2\log r}\right) \to \operatorname{sign}(\alpha)\infty$$

unless $\alpha = 0$. Thus averages on circles of u go like $0 \log r + 0$. Since the α is the period of the harmonic function, it must be harmonic in the whole disc, and thus by the maximum principle is a constant zero.

Problem 5

Let $\{f_n\}$ be holomorphic functions on $\mathbb{D}$ and suppose $\|f_n\|_{L^1} \le 1$ for all n. Show that this is a normal family.

Better Proof

Let's show that the family is uniformly bounded. We do the usual trick to convert between L^1 information and L^∞ information: the mean value property.

Let $r < 1$. For $z \in B(0, r)$, we have

$$\begin{aligned} |f_n(z)| &\le \frac{1}{\pi(1-r)^2} \int_{B(z,1-r)} f_n(z) \\ &\le \frac{1}{\pi(1-r)^2} \end{aligned}$$

which is a locally uniform bound. Apply Montel's theorem.

Proof

Note that $f_n(z) = \fint_{B(z,r)} f_n \, d\lambda(z)$ where λ is the Lebesgue measure. This gives a uniform bound on compact subsets. Note that for analyic functions, this is enough for equicontinuity as well. Thus Arzela–Ascoli gives us that this family is normal.

Proof of equicontinuity:

$$
\begin{aligned}
f(z) - f(z_0) &= \fint_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right) f(\zeta) \, d\zeta \\
&= \frac{z - z_0}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)} \, d\zeta
\end{aligned}
$$

Thus if $|f| \leq M$ on C and $z, z_0 \in B(0, r/2)$, then this is all bounded by

$$4M|z - z_0|/r$$

which shows locally uniformly Lipschitz (not just equicontinuity!).

Problem 6

Let $0 \in U \subseteq \mathbb{C}$ be open and bounded. Let $f : U \to U$ be holomorphic with $f(0) = 0$. Show that $|f'(0)| \leq 1$.

Proof

Let's consider $f^n = f \circ \cdots \circ f$. By the chain rule, $(f^n)' = (f')^n$, so if $|f'(0)| > 1$, then $g = f^n$ has arbitrarily large derivative at 0 with $g(0) = 0$

Write $|g(z)| \leq R$. The Cauchy integral formula gives us

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int \frac{f(z)}{z^{n+1}} \, dz$$

Thus $|g'(0)| \leq Rr/r^2$ for some $B(0, r) \subseteq U$. But this uniform bound is a contradiction. Thus $|f'(0)| \leq 1$.

Problem 7

Show that there is a dense set of functions $f \in L^2([0, 1])$ such that $x \mapsto x^{-1/2} f(x) \in L^1$ and $\int_0^1 x^{-1/2} f(x) \, dx = 0$.

Proof

Let's try to find functions with small norm in L^2 that we can use to get density but whose L^1 norm against $1/\sqrt{x}$ is some fixed quantity (maybe 1).

We define $w_n = \frac{1}{n} x^{\frac{-1}{2}+\frac{1}{n}}$. Note that these are in L^2, are in L^1 when multiplied by $x^{-1/2}$. These are easy to check.

Furthermore, the integrals are all 1 against $x^{-1/2}$, and the L^2 norms go to zero.

Then consider $S = \{g - cw_n \mid g \in C([0,1]), c = \int_0^1 x^{-1/2} g\}$. For each $f \in S$, the integral against $x^{-1/2}$ is zero.

First of all, $S \subseteq L^2$ and $x^{-1/2} S \subseteq L^1$. The integrals against $x^{-1/2}$ are all zero.

Finally, density. Let $f \in L^2$ and let $g \in C([0,1])$ be close to f in L^2. Picking a w with very small L^2 norm gives us the desired result.

Problem 8

Compute the following limits:

$$\lim_{k\to\infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k \, dx$$

where $n \in \mathbb{N}$ and

$$\lim_{k\to\infty} \int_0^\infty \left(1 + \frac{x}{k}\right)^{-k} \cos(x/k) \, dx$$

Proof, Part 1

Recall that $(1+1/n)^n \to e$. By messing around, we know that $(1-x/k)^k \to e^{-x}$. Let's show it's monotone convergence so we can apply the monotone convergence theorem.

We can either try to optimize $(1-x/k)^k/(1-x/(k+1))^{k+1}$ and show it's less than 1 always or apply the AM-GM inequality cleverly:

$$\begin{aligned}\left(1^1 \cdot \left(1 - \frac{x}{k}\right)^k\right)^{1/(k+1)} &\leq \frac{1 + k(1 - x/k)}{k+1} \\ &= 1 - \frac{x}{k+1}\end{aligned}$$

and so $(1 - x/k)^k \leq (1 - x/(k+1))^{k+1}$ so we have monotonicity.

Thus the integral converges to $\Gamma(n+1) = n!$ by the monotone convergence theorem.

Proof, Part 2

We are concerned about cos being not integrable, but having only an easy L^∞ bound.

Our AM-GM argument flips now: $(1 + x/k)^{-k} \leq (1 + x/2)^{-2}$. This gives us the integrability we need to deal with cos, since now $(1 + x/2)^{-2}\cos(x/k)$ is integrable, so we apply dominated convergence to send $\cos(x/k) \to 1$ and $(1 + x/k)^{-k} \to e^{-x}$ and overall get $\int e^{-x} = 1$.

Alternatively, consider the following argument.

We need some x^{-2} behavior to kill boundedness of cosine to get something integrable. Let's expand

$$(1 + x/k)^k = 1 + k(x/k) + \binom{k}{2}(x/k)^2 + \cdots \geq \frac{2 + x^2}{2}$$

It's easy to apply dominated convergence now and then take a limit on the inside and get $\int_0^\infty e^{-x} = 1$.

Problem 9

Let X be Banach, Y an NVS, and $B : X \times Y \to \mathbb{R}$ be bilinear. Suppose that for each $x \in X$, there exists $C_x \geq 0$ such that $|B(x, y)| \leq C_x\|y\|$ and for each y there is a constant $C_y \geq 0$ such that $|B(x, y)| \leq C_y\|x\|$ for all $x \in X$.

Show that $|B(x, y)| \lesssim \|x\|\|y\|$.

Proof

Just define $B_y(x) = B(x, y)$ for each $\|y\| = 1$. Then each $\left\|B_y(x)\right\| \leq C_x$ for each x. The uniform boundedness principle says that $\left\|B_y\right\| \leq C$ for every $\|y\| = 1$. This is all we need.

Problem 10

1. Let $f \in L^2(\mathbb{R})$ and define $h(x) = f * f = \int f(x - y)f(y)\,dy$. Show that there exists $g \in L^1$ such that $h(\xi) = \int e^{-i\xi x}g(x)\,dx$, i.e., $h = \hat{g}$.

2. Conversely show that if $g \in L^1$ then there is a function $f \in L^2(\mathbb{R})$ such that the Fourier transform of g is given by $h = f * f$.

Proof, Part 1

Recall that $F^{-1}(f * g) = F^{-1}f \cdot F^{-1}g$. Thus $g = F\left[\left(F^{-1}f\right)^2\right]$ works.

We need to rely on the fact that $f * f$ and $(F^-1f)^2$ are both continuous to apply Fourier inversion. The theorem says continuous and L^1 implies Fourier inversion holds. Products of L^2 functions are L^1 so we're good.

Alternatively, $\phi, \hat{\phi} \in L^1$ implies Fourier inversion for ϕ, which is easier.

Proof, Part 2

If $g \geq 0$, then $\sqrt{g} \in L^2$, so $F^{-1}(\sqrt{g}) \in L^2$ and $f = F^{-1}\sqrt{g}$ works. If $g \geq 0$ isn't true, maybe consider complex valued functions and take a branch cut of the square root.

Problem 11

Consider $C([0,1])$ with the usual norms. Let S be a subspace. Show that if there exists $K \geq 0$ such that $\|f\|_{L^\infty} \leq K\|f\|_{L^2}$ for all $f \in S$, then S is finite-dimensional.

Proof

Let $f_n \in S$ and $f_n \to f$ in L^2. Then f_n has an L^∞ limit to a function in $C([0,1])$ because they're continuous and Cauchy in the sup norm.

Thsi doesn't actually prove closedness, but instead closedness of the L^2-closure of S.

However, the property of $\|f\|_\infty \leq K\|f\|_{L^2}$ is preserved under L^2 limits. That is, if $f_n \to f$ in L^2 and $\|f_n\|_{L^\infty} \leq K\|f_n\|_{L^2}$, then $\|f_m - f_n\|_{L^\infty} \leq K\|f_m - f_n\|_{L^2}$, so $\|f_n\|_{L^\infty} \to \|f\|_{L^\infty}$ and the inequality at f holds.

Thus S may be replaced with its \$L^2\$-closure, and can thus be considered a Hilbert space.

Consider the evaluation maps $e_x : S \to \mathbb{R}$ given by $e_x(f) = f(x)$. These are L^∞ bounded and thus L^2 bounded operators. These are continuous too, so they have duals g_x with $\|g_x\|_{L^2} \leq K$. Let's look at the norm squared by comparing with an ONB.

$$K^2 \geq \|g_x\|^2 = \sum |\langle f_i, g_x \rangle|^2 = \sum |f_i(x)|^2$$

Integrating in x tells us $\sum 1 \leq K^2$, so $\dim \overline{S} \leq K^2$.

Problem 12

Let $f : [0,1] \to \mathbb{R}$ be continuous which is ac on each $[\varepsilon, 1]$ with $0 < \varepsilon \leq 1$. Show that f is not necessarily ac on $[0,1]$. Show that if f is bv on $[0,1]$ then f is ac on $[0,1]$.

Proof, Part 1

First, look at $x^2 \sin(1/x^2)$. Since f' is bounded on $[\varepsilon, 1]$, it's ac. An explicit bound is $1/\varepsilon + 2$. Thus it's Lipschitz on $[\varepsilon, 1]$, so it's ac on $[\varepsilon, 1]$.

Note: ac implies bv, so it's enough to show that f is not bv on $[0,1]$.

Let's consider intervals

$$a_n = \left(\frac{(2n-1)\pi}{2}\right)^{-1/2}$$
$$b_n = \left(\frac{(2n+1)\pi}{2}\right)^{-1/2}$$

Then

$$f(b_n) = \left(\frac{2}{(2n+1)\pi}\right)$$
$$f(a_n) = -\left(\frac{2}{(2n-1)\pi}\right)$$

so the difference grows like $\sum 1/n \to \infty$. Thus f can't be bv, because here we have some variation which goes to infinity.

Proof, Part 2

For the next part, if f is bv also, then there is some total variation T_f.

If you want f to be ac, pick η small such that $T_f(x) < \varepsilon/2$ on $[0, \eta]$, since T_f is increasing (and continuous!). Then find $\delta > 0$ for ac with $\varepsilon/2$ for the rest of the interval, and near the origin we bound by the total variation on $[0, \eta]$, which is less than $\varepsilon/2$. Formally:

$$\sum |f(b_n) - f(a_n)| \leq \sum_{b_j \leq \eta} \left| f(b_j) - f(a_j) \right| + \sum_{a_j \geq \eta} < T_f(\eta) + \varepsilon/2 < \varepsilon$$

where we refine the partition maybe to split it like that.

Just to cover our bases, recall that f is BV iff $f = g - h$, two monotone functions. Then $T_f = g + h$ which is continuous (since if $g + h$ is discontinuous, then either g or h is, and since f is continuous, their jump discontinuities must cancel, so we can WLOG get rid of them).

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Problem 1

Consider a measure space $(X, \mathcal{X})$ with a σ-finite measure μ and for each $t \in \mathbb{R}$ let $e_t = 1_{(t,\infty)}$ be the characteristic function. Prove that if $f, g : X \to \mathbb{R}$ are $\mathcal{X}$ measurable, then

$$\|f - g\|_{L^1(X)} = \int_{\mathbb{R}} \|e_t \circ f - e_t \circ g\|_{L^1(X)}\, dt$$

Proof

Without loss of generality, $f \geq g$ everywhere. Suppose we have the result in this case, then for general f and g, write $F = f \vee g$ and $G = f \wedge g$. Then $\|F - G\| = \|f - g\|$ and $\|e_t \circ f - e_t \circ g\| = \|e_t \circ F - e_t \circ G\|$ so the result follows.

Now we assume $f \geq g$. Note that $\|e_t \circ f - e_t \circ g\|_{L^1(X)} = m(\{f > t > g\})$.

Let's write

$$\begin{aligned}
\|f - g\|_{L^1(X)} &= \int_X f(x) - g(x)\, d\lambda(x) \\
&= \int_X \int_{g(x)}^{f(x)} 1\, dt\, d\lambda(x) \\
&= \int_X \int_{\mathbb{R}} 1_{(g(x), f(x))}(t)\, dt\, d\lambda(x) \\
&= \int_{\mathbb{R}} \int_X 1_{(g(x), f(x))}(t)\, d\lambda(x)\, dt
\end{aligned}$$

by Tonelli and non-negativity. We evaluated the inner integral to be $m(\{g(x) < t < f(x)\})$, i.e., $\|e_t \circ f - e_t \circ g\|_{L^1(X)}$.

Problem 2

Let $f \in L^1(\mathbb{R}, dx)$ and $\beta \in (0, 1)$. Prove that

$$\int_{\mathbb{R}} \frac{|f(x)|}{|x - a|^{\beta}}\, dx < \infty$$

for Lebesgue a.e. $a \in \mathbb{R}$.

Proof

We verify local integrability in a. First we inspect the multiplier:

$$\begin{aligned}\int_n^{n+1} \frac{1}{|x-a|^\beta} da &\leq \int 2 \int_0^1 \frac{1}{t^\beta} \\ &= \frac{2}{1-\beta}\end{aligned}$$

where we just shift the denominator by enough to be contained in $(-1, 1)$.

Then we apply Tonelli

$$\begin{aligned}\int_n^{n+1} \int_{\mathbb{R}} |f(x)| \frac{1}{|x-a|^\beta} \, dx \, da &= \int_{\mathbb{R}} |f(x)| \int_n^{n+1} \frac{1}{|x-a|^\beta} \, da \, dx \\ &\leq C\|f\|_{L^1}\end{aligned}$$

Problem 3

Let $[a, b] \subseteq \mathbb{R}$ be a finite interval and let $f : [a, b] \to \mathbb{R}$ be a bounded Borel measurable function.

1. Prove that $\{x \in [a, b] \mid f \text{ is continuous at } x\}$ is Borel measurable.

2. Prove that f is Riemann integrable if and only if it is continuous almost everywhere.

Proof, Part 1

It's a well-known fact that the set of continuity points of a (reasonable) function are G_δ. Let's consider the following (relatively) open sets:

$$E_n = \{x \in [a, b] \mid \exists \varepsilon > 0. \ \forall y, z \in B(x, \varepsilon). \ |f(y) - f(z)| < 1/n\}.$$

Then $\bigcap E_n$ is the set of continuity points of f in $[a, b]$.

Proof, Part 2

Let's show that f is Riemann integrable iff it is continuous almost everywhere.

First, let A be the upper Riemann integral and B the lower Riemann integral. Then there are nested partitions $P_1 \subseteq P_2 \subseteq \ldots$ of $[a, b]$ such that (where $I_{n,m}$ is the m-th interval in the partition P_n)

$$\sum_m \sup_{I_{n,m}} f \cdot |I_{n,m}| =: U(f, P_n) \searrow A$$
$$\sum_m \inf_{I_{n,m}} f \cdot |I_{n,m}| =: L(f, P_n) \nearrow B$$

as $n \to \infty$.

Define $U_n = \sum_m \sup_{I_{n,m}} f \cdot \chi_{I_{n,m}}$ and L_n analogously. Note that $L_1 \leq L_2 \leq \cdots \leq f \leq \cdots \leq U_2 \leq U_1$.

Note that $L_n \nearrow L$ and $U_n \searrow U$ with integrals $\int L = B$ and $\int U = A$ by dominated convergence.

Now finally we have that $A = B$ iff $\int U - L = 0$, i.e., $U - L = 0$ almost everywhere.

If $U(x) = L(x)$, then (ignoring x that are ever an endpoint of an interval in a selected partition) we have that f is continuous at x. This is the desired result.

Problem 4

1. Consider a sequence $\{a_n, n \geq 1\} \subseteq [0, 1]$. For $f \in C([0, 1])$ let us denote

$$\phi(f) = \sum_{n=1}^{\infty} \frac{f(a_n)}{2^n}.$$

   Prove that there is no $g \in L^1([0, 1])$ such that $\phi(f) = \int f(x)g(x)\,dx$ for all $f \in C([0, 1])$.

2. Each $g \in L^1([0, 1])$ defines a continuous functional T_g on $L^\infty([0, 1])$ by

$$T_g(f) = \int f(x)g(x)\,dx.$$

   Show that there are continuous functionals on $L^\infty([0, 1])$ that are not of this form.

Proof, Part 1

Suppose there is such a g. Let f_k be a piecewise linear function with support $[a_1 - 1/k, a_1 + 1/k]$ which is 1 at a_1 and linear in between.

Then $\phi(f_k) \geq 1/2$. However, $\int f_k g \to 0$ by dominated convergence.

Proof, Part 2

Consider $\lim_{x \to 1} f(x)$ on the set of functions where the limit exists. Hit with Hahn–Banach to extend (since it's obviously continuous).

It's easy to see this can't be induced by an integrable function.

Alternatively, assume for contradiction every $T \in (L^\infty)^*$ is T_g for some $g \in L^1$. Then $g \mapsto T_g$ is an isomorphism. It's surjective by hypothesis and injective obviously. It's also bounded because

$$\left\|T_g\right\|_{\text{op}} = \sup_{\|f\|_{L^\infty} \leq 1} \left| \int_0^1 fg \right| \leq \left| \int_0^1 g \right| \leq \|g\|_{L^1}$$

so by the open mapping theorem, its inverse is bounded as well.

However, L^1 is separable, but then $(L^\infty)^*$ is separable, so L^∞ is as well. This is a contradiction! Note that $\{\chi_{[0,r]} \mid 0 < r < 1\}$ is an uncountable discrete set in L^∞.

Problem 5

Recall that a metric space is separable if it contains a countable dense subset.

1. Prove that $\ell^1(\mathbb{N})$ and $\ell^2(\mathbb{N})$ are separable Banach spaces, but $\ell^\infty(\mathbb{N})$ is not.
2. Prove that there exists no linear bounded surjective map $T : \ell^2(\mathbb{N}) \to \ell^1(\mathbb{N})$.

Proof, Part 1

Let's show these spaces are separable. Note that we have the containment $\ell^1 \subseteq \ell^2$, so it's enough to show that ℓ^2 is separable using a countable dense subset of summable sequences.

Let's consider the $\mathbb{Q}$-span of $\{e_i\}$, i.e., sequences with finitely man non-zero terms, all of which are rational. It's fairly trivial to show these are dense.

On the other hand, consider all binary sequences. There are uncountably many, all of which have L^∞ distance at least 1 away, so L^∞ isn't separable.

Proof, Part 2

If $T : \ell^2 \to \ell^1$ were bounded surjective and linear, we'd have a bounded injective linear map $T^* : \ell^\infty \to \ell^2$, so we could identify a non-separable linear subspace of ℓ^2, a contradiction.

Problem 6

Given a Hilbert space H, let $\{a_n\}_{n\geq 1} \subseteq H$ be a sequence with $\|a_n\| = 1$ for all $n \geq 1$. Recall that the closed convex hull of $\{a_n\}_{n\geq 1}$ is the closure of the set of all convex combinations of elements in $\{a_n\}$.

1. Show that if $\{a_n\}_n$ spans H linearly (i.e., any $x \in H$ is a finite linear combination of the $\{a_n\}$), then H is finite dimensional.
2. Show that if $\langle a_n, \xi \rangle \to 0$ for all $\xi \in H$, then 0 is in the closed convex hull of $\{a_n\}_n$.

Proof, Part 1

Write $H = \bigcup \operatorname{span}\{a_1, \ldots, a_n\}$. Each is nowhere dense, but the union is a Banach space, a contradiction with the Baire category theorem.

Proof, Part 2

Fix $\varepsilon > 0$. Let's show there's a convex combination of the a_n with norm less than ε. First, pick $a_{N_1} = a_1$. Then pick a_{N_2} such that $\left|\left\langle a_{N_2}, a_{N_1}\right\rangle\right| < \varepsilon$. Inductively create a sequence a_{N_k} such that $\left|\left\langle a_{N_j}, a_{N_k}\right\rangle\right| < \varepsilon$ for all j, k.

Pick $\mathbb{N} \ni r > 1/\varepsilon$ and consider $v = (1/r)a_{N_1} + \cdots + (1/r)a_{N_r}$. We measure the norm:

$$\begin{aligned}\left\|\frac{1}{r}a_{N_1} + \cdots + \frac{1}{r}a_{N_r}\right\|^2 &= \frac{1}{r^2}\left\langle a_{N_1} + \cdots + a_{N_r}, a_{N_1} + \cdots + a_{N_r}\right\rangle \\ &\leq \frac{1}{r^2}(r + r^2\varepsilon) < 2\varepsilon\end{aligned}$$

Problem 7

Characterize all entire functions f with $|f(z)| > 0$ for $|z|$ large and

$$\limsup_{z\to\infty} \frac{\left|\log|f(z)|\right|}{|z|} < \infty$$

Proof

First, note that there are finitely many zeroes to f. Let p share these zeroes and be a polynomial. Then we consider $f/p =: g$ which has no zeroes and

$$\frac{\left|\log|g(z)|\right|}{|z|} = \frac{\left|\log|f(z)| - \log|p(z)|\right|}{|z|} < \infty$$

so g satisfies the same conditions.

Now that g has no zeroes, let's write it as $\exp(h)$ for some h. Then

$$\limsup_{z\to\infty} \frac{\left|\log\left|e^{h(z)}\right|\right|}{|z|} = \limsup_{z\to\infty} \frac{|\operatorname{Re} h(z)|}{|z|}$$

Thus $|\operatorname{Re} h(z)| \leq C|z|$ for some C and all large z.

We claim that h is a degree 1 polynomial. If it were $|h(z)|$ instead of the real part, we could apply Cauchy's integral formula (see "Entire function bounded by a polynomial is a polynomial" in list.pdf) to conclude that $h(z)$ is a degree 1 polynomial.

Alternatively, we can shift h to have a zero at the origin and then h/z does not reach arbitrarily large real numbers, and hence is constant by another qual problem.

Instead we just consider $h = u + iv$. Write $h(re^{i\theta}) = \sum a_n r^n e^{in\theta}$ and we can compute

$$\int_0^{2\pi} u(re^{i\theta})e^{-ik\theta}\, d\theta = \pi r^k a_k$$

so that

$$|a_k|r^k \leq \frac{1}{\pi}\int_0^{2\pi} \left|u(re^{i\theta})\right| d\theta$$

Similarly, we can compare with the mean value property to get

$$|a_k|r^k + 2u(0) \leq \frac{1}{\pi}\int_0^{2\pi} |u| + u\, d\theta \lesssim Cr$$

and so $|a_k| = 0$ for all $k > 1$ by sending $r \to \infty$. Thus u is a polynomial of degree 1.

Thus $f(z) = p(z)e^{az+b}$ for some polynomial p and $a, b \in \mathbb{C}$.

Problem 8

Construct a non-constant entire function $f(z)$ such that the zeroes of f are simple and coincide with the set of all (positive) natural numbers.

Proof

We'd like to write

$$\prod_{n>0}\left(1-\frac{z}{n}\right)$$

but this product does not converge.

Recall that Weierstrass products introduce an exponential term to enforce convergence. Recall that the genus of a product with prescribed zeroes a_n is the smallest h such that $\sum 1/|a_n|^{h+1}$ converges.

Clearly this product will have genus 1. By general facts about genus, the following product converges:

$$\prod_{n>0}\left(1-\frac{z}{n}\right)e^{z/n}$$

This clearly has the right number of zeroes of the right order at the right points. To double check convergence:

$$\sum_{n>0}\log|1-(z/n)|+\log\left|e^{z/n}\right| = \sum_{n>0}\log(1-z/n)+z/n$$

Note that $\log(1-z/n)+z/n = \frac{1}{2}(z/n)^2+\dots$ so for z in a compact set, eventually we have a convergent series, so this is a locally uniform and absolute convergence.

Problem 9

Prove Hurwitz's Theorem: Let $\Omega \subseteq \mathbb{C}$ be a connected open set and $f_n, f : \Omega \to \mathbb{C}$ holomorphic functions. Assume that $f_n \to f$ locally uniformly. Prove that if $f_n \neq 0$ everywhere in Ω, then either f is identically equal to 0 or $f(z) \neq 0$ anywhere in Ω.

Proof

Suppose $f_n(a) = 0$. Either there is a punctured neighborhood on which f is nonzero, or f is identically zero.

In the former case, consider $r > 0$ so small that $f_n(z) \neq 0$ for $0 < |z-a| < 2r$. Let γ be a circle of radius r around a. Then, $n(f\circ\gamma, 0) = 1$.

But since the f_n all lack zeroes, $n(f_n \circ \gamma, 0) = 0$. And by uniform convergence, these winding numbers must converge to 1. This is a contradiction.

To show that the winding numbers converge, we write

$$n(f \circ \gamma, 0) = \frac{1}{2\pi i} \int_{C_r} \frac{f'}{f} \, dz$$

and note that on C_r, $1/f_n \to 1/f$ uniformly as well as $f_n' \to f'$ uniformly. This is the argument principle.

Problem 10

Let $\alpha \in [0,1] \setminus \mathbb{Q}$ and let $\{a_n\} \in \ell^1(\mathbb{N})$ with $a_n \neq 0$ for all $n \geq 1$. Show that

$$f(z) = \sum_{n \geq 1} \frac{a_n}{z - e^{i\alpha n}}$$

for $z \in \mathbb{D}$ converges and defines a function that is analytic in $\mathbb{D}$ which does not admit an analytic continuation to any domain larger than $\mathbb{D}$.

Proof

Let's prove convergence. Let $|z| = r$.

$$\begin{aligned} |f(z)| &\leq \sum_{n \geq 1} \frac{|a_n|}{|z - e^{i\alpha n}|} \\ &\leq \|a_n\|_{\ell^1} \frac{1}{1-r} \end{aligned}$$

This is absolute convergence, and in $B(0,r)$, we have the tail of the sum goes to zero uniformly (since it's bounded by $1/(1-r) \sum_{n>k} |a_n|$).

To show that this does not admit an analytic continuation, first we recall that $e^{i\alpha n}$ is dense in the circle.

It's hard to show that f blows up at $e^{i\alpha n}$ for each n. But we can consider instead $g(z) = (z - e^{i\alpha n}) f(z)$ where f is the analytic continuation and $e^{i\alpha n}$ is contained in the expanded region.

Note that $g(e^{i\alpha n}) = 0$. Let's write

$$g(re^{i\alpha n}) = a_n + \sum_{k \neq n} \frac{a_k (r-1) e^{i\alpha n}}{re^{i\alpha n} - e^{i\alpha k}}$$

Now let's bound

$$\left| \frac{a_k (r-1) e^{i\alpha n}}{re^{i\alpha n} - e^{i\alpha k}} \right| \leq |a_k|$$

and so by dominated convergence (with the sum), we know that $g(re^{i\alpha n}) \to a_n \neq 0$.

This is a contradiction!

Problem 11

For each $p \in (-1, 1)$ compute the improper Riemann integral

$$\int_0^\infty \frac{x^p}{x^2+1}\,dx$$

Proof

Let's do a key-hole contour. First, select the branch cut of log by cutting along the positive reals. We pick a contour that goes from $r + i\varepsilon$ to $R + i\varepsilon$ (where $r \ll 1 \ll R$) then CCW along a circle to $R - i\varepsilon$, then back to $r - i\varepsilon$, the CW along a circle of radius r to $r + i\varepsilon$. Denote the circlular arcs c_r and C_R.

The integral on the small circle is bounded by

$$\int_{c_r} \frac{z^p}{z^2+1}\,dz \leq \frac{r^{1-p}}{r^2+1} \to 0$$

since $p < 1$. Similarly, the integral on the large circle goes to zero as $r \to \infty$.

Thus we're left with

$$(1 + \exp(2p\pi i)) \int_0^\infty \frac{x^p}{x^2+1} = 2\pi i \operatorname{Res}_i f + 2\pi i \operatorname{Res}_{-i} f$$

After computing the residue and a lot of rearranging, we get $\pi/(2\cos(\pi p/2))$.

Problem 12

Compute the number of zeros, including multiplicity, of $f(z) = z^6 + iz^4 + 1$ in the upper half plane in $\mathbb{C}$.

Proof

Note that if z is a zero, so is $-z$. Furthermore, there are clearly no real zeroes, since 0 is not one, and if $z \in \mathbb{R}$, then $z^6 \in \mathbb{R}$ but $iz^4 + 1$ is definitely not.

Thus half of them are in the upper half-plane, i.e., 3 zeroes in the upper half plane.

6 F14

Problem 1

Show that

$$A = \{f \in L^3(\mathbb{R}) \mid \int_{\mathbb{R}} |f|^2\,dx < \infty\}$$

is a Borel subset of $L^3(\mathbb{R})$.

Proof

Write

$$A = \bigcup_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \left\{ f \in L^3 : \int_{-n}^{n} |f|^2 < m \right\}.$$

This is natural because on each finite interval, $L^3 \subseteq L^2$. Let's show that $\{f \in L^3 \mid \int_{-n}^{n} |f|^2 < m\} = B_{n,m}$ is Borel. In particular, let's show it's open.

This is easy, because $\|f\|_{L^2([-n,n])} \lesssim \|f\|_{L^3([-n,n])}$, so for $f \in B_{n,m}$, take $r \lesssim m - \|f\|_{L^2([-n,n])}$. Then $B_{L^3}(f, r) \subseteq B_{n,m}$. The constant depends on n.

Problem 2

Construct an $f \in L^1(\mathbb{R})$ so that $f(x+y)$ does *not* converge almost everywhere to $f(x)$ as $y \to 0$. Prove that your f has this property.

Proof

Why not just take $f = \chi_{\mathbb{Q}}$? I mean technically $f = 0$, but by writing $f(x+y)$, we're already taking a specific representative.

Alternatively, take a fat Cantor set A and consider χ_A. At every point in A, there are arbitrarily close points not in A.

Problem 3

Let (f_n) be a bounded sequence in L^2 and suppose $f_n \to 0$ Lebesgue almost everywhere. Show that $f_n \rightharpoonup 0$ weakly in L^2.

Proof

Let $g \in L^2$ and $\varepsilon > 0$. Pick $R > 0$ such that $\int_{[-R,R]^c} |g|^2 < \varepsilon$.

Note that convergence almost everywhere on a finite measure space implies convergence in measure (technically the result is that on a σ-finite space, ae implies *locally* in measure, but if the space has finite measure, then it is *global* too).

Thus on $[-R, R]$, we have convergence in measure of $f_n g \to 0$. Therefore we can consider Vitali's convergence theorem. All we need is uniform integrability and tightness to show that $f_n g \to 0$ in L^1.

First, uniform integrability, i.e., for all $\varepsilon > 0$ there exists $\delta > 0$ such that $m(A) < \delta$ implies $\int_A |f_n g| < \varepsilon$.

(This and Egorov actually are enough, and this uniform integrability is shown in the second proof.)

Now for tightness, i.e., for all $\varepsilon > 0$ there exists E such that $\int_{E^c} |f_n g| < \varepsilon$ for all n. Basically the same argument works. It actually follows trivially from uniform integrability on a finite measure space ($[-R, R]$).

(Different motivation) Proof

Let $g \in L^2$ and $\varepsilon > 0$. Pick $R > 0$ such that $\int_{[-R,R]^c} |g|^2 < \varepsilon$. Pick $E \subseteq [-R, R]$ such that $m([-R, R] \setminus E) < \delta$ and $f_n \to 0$ uniformly on E by Egorov's theorem where we pick δ later.

Let $\|f_n\|_{L^2} \le M$ be the uniform bound. Then

$$\begin{aligned}\left|\int f_n g\right| &\le \left|\int_{[-R,R]^c} f_n g\right| + \left|\int_{[-R,R]\setminus E} f_n g\right| + \left|\int_E f_n g\right| \\ &\le \|f_n\|_{L^2}\varepsilon + \cdots + \|f_n\|_{L^\infty(E)}\|g\|_{L^1} \\ &\lesssim \|f_n\|_{L^2}\varepsilon + \cdots + \|f_n\|_{L^\infty(E)}\|g\|_{L^2}\end{aligned}$$

The problem is the middle term. Sure, the integral of an L^1 function is small on a small set, but this needs to be independent of n.

However, this can be guaranteed! Since $|g|^2 \in L^1([-R, R])$, there exists $\delta > 0$ such that if $m(A) < \delta$, then $\int_A |g|^2 < \varepsilon^2$.

Then if $m(A) < \delta$, we have that $\int_A |f_n g| \le M\varepsilon$ by Cauchy-Schwarz.

We now identify the middle term as being less than $M\varepsilon$ as desired. Send $\varepsilon \to 0$ and we're done.

Easier Proof

By Banach-Alaoglu, for every subsequence f_{n_k}, there is a further subsequence converging to something in L^2. This limit must be zero, so in fact $f_n \to 0$.

For clarification on why the limit must be zero, suppose $f_n \rightharpoonup f$ and $f_n \to 0$ almost everywhere. Consider a set A where $f \ge 0$ but with finite measure, and then apply Egorov to take a large subset. On this large subset, $f = 0$.

Problem 4

Given $f \in L^2([0,\pi])$, we say that $f \in \mathscr{G}$ if f admits a representation of the form

$$f(x) = \sum_{n=0}^{\infty} c_n \cos(nx) \text{ with } \sum_{n=0}^{\infty}(1+n^2)|c_n|^2 < \infty$$

show that if $f \in \mathscr{G}$ and $g \in \mathscr{G}$, then $fg \in \mathscr{G}$.

Proof

Note that this is describing the Sobolev space H^1 (but with Fourier series), so we expect fg to have a derivative, i.e., $f'g + fg'$, where hopefully $f'g, fg' \in H^1$.

First, let's try to use the usual Fourier series. Let's identify $L^2([0,\pi])$ with $L^2_e([-\pi,\pi])$, the subspace of even functions.

Recall that $\{\cos(nx)\}$ is an ONB for L^2_e (since usually we need $\{\cos(nx), \sin(nx)\}$, but for even functions all we need is cosine.

For $f \in \mathscr{G}$, note that $\sum |c_n| < \infty$ (a quick inequality shows this) so the series representation converges uniformly and absolutely on $[-\pi,\pi]$.

Take L^2_e with inner product $\langle f, g\rangle = ⨍_{-\pi}^{\pi} fg$. Next, we note $\langle f, \cos(nx)\rangle = c_n/2$ for $n \neq 0$ and c_0 for $n = 0$, since $⨍_{-\pi}^{\pi} \cos^2(nx) = 1/2$, but $⨍ 1 = 1$.

Similarly, we can calculate with Fourier series that $\langle f, \cos(nx)\rangle = \frac{1}{2}(\hat{f}(n) + \hat{f}(-n)) = \hat{f}(n)$ because f is even (and even functions have even Fourier series).

Thus $c_n = 2\hat{f}(n)$ for $n \neq 0$ and $c_0 = \hat{f}(0)$. It suffices then to show that $\langle n\rangle \widehat{fg} \in \ell^2$, where $\langle n\rangle = \sqrt{1+n^2}$ is the Japanese bracket.

First, note that $\langle n\rangle \lesssim \langle n-k\rangle + \langle k\rangle$. This is with analogy to the triangle inequality and takes some annoying calculation to show.

We calculate

$$\begin{aligned}
\langle n\rangle \widehat{fg}(n) &= \sum_k \langle n\rangle \hat{f}(k)\hat{g}(n-k) \\
&\lesssim \sum_k \left(\langle k\rangle + \langle n-k\rangle\right) \hat{f}(k)\hat{g}(n-k) \\
&= \sum_k \langle k\rangle \hat{f}(k)\hat{g}(n-k) + \sum_k \langle n-k\rangle \hat{g}(n-k)\hat{f}(k) \\
&\leq \left[\left(\langle\cdot\rangle\hat{f}\right) * \hat{g}\right] + \left[(\langle\cdot\rangle\hat{g}) * \hat{f}\right] \\
\left\|\langle n\rangle \widehat{fg}(n)\right\|_{\ell^2} &\leq \|\langle\cdot\rangle\hat{f}\|_{\ell^2}\|\hat{g}\|_{\ell^1} + \|\langle\cdot\rangle\hat{g}\|_{\ell^2}\|\hat{f}\|_{\ell^1}
\end{aligned}$$

Note that $\sum|c_n| \leq \sum |c_n|\langle n\rangle\langle n\rangle^{-1} \lesssim \||c_n|\langle n\rangle\|_{\ell^2}$, so $\hat{f}$ and $\hat{g}$ are in ℓ^1 and everything's finite.

Problem 5

Let $\phi : [0,1] \to [0,1]$ be continuous and let μ be a Borel probability measure on $[0,1]$.

Suppose $\mu(\phi^{-1}(E)) = 0$ for every Borel sets $E \subseteq [0,1]$ with $\mu(E) = 0$. Show that there is a Borel measurable function $w : [0,1] \to [0,\infty)$ such that

$$\int f \circ \phi(x)\, d\mu(x) = \int f(y)w(y)\, d\mu(y)$$

for all continuous $f : [0,1] \to \mathbb{R}$.

Proof

We write $\int f \circ \phi\, d\mu(x) = \int f\, d\phi_*\mu$ where $\phi_*\mu(E) = \mu(\phi^{-1}(E))$ (where this is a measure because ϕ is continuous and hence measurable).

We thus want to write $\int f\, d\phi_*\mu = \int f w\, d\mu$ for some w.

This looks like Radon-Nikodym. We need to verify that $\phi_*\mu \ll \mu$. But if $\mu(E) = 0$ then $\phi_*\mu(E) = 0$ by hypothesis.

Thus $\phi_*\mu(E) = \int_E w\, d\mu$ for some integrable (and Borel and positive) function w.

The equality $\int f\, d\phi_*\mu = \int f w\, d\mu$ holds trivially for simple functions. By density in L^1 we can find a simple function to approximate our arbitrary continuous function in both $L^1(\phi_*\mu)$ and $L^1(\mu)$ (since normally our simple functions increase up to the continuous function, we just take a maximum).

Problem 6

Let X be a Banach space and X^* its dual space. Suppose X^* is separable. Show that X is separable. (You should assume the Axiom of Choice.)

Proof

Let $\{\alpha_n\}$ be a countable dense subset of X^*. For each α_n select x_n such that $\alpha_n(x_n) > (1/2)\|\alpha_n\|$ and $\|x_n\| = 1$ (thanks AoC!) and consider all $\mathbb{Q}$-linear combinations of the x_n.

I claim that $A = \operatorname{span}_{\mathbb{Q}}\{x_n\}$ is dense in X. Suppose it isn't! Take the closure $\overline{A}$ (which is obviously a closed subspace). Suppose $v \notin \overline{A}$ and $\|v\| = 1$.

Consider $B = \operatorname{span}\{v\}$ and define $\alpha(v) = 1$ a functional on B. This is a continuous functional (being on a one-dimensional subspace), so we can extend it by Hahn-Banach to get a functional which is zero on $\overline{A}$. Now consider $\alpha_n \to \alpha$

$$\|\alpha - \alpha_n\| \geq |\alpha_n(x_n) - \alpha(x_n)| = |\alpha_n(x_n)| > \frac{1}{2}\|\alpha_n\|$$

so $\|\alpha_n\| \to 0$ so $f = 0$ a contradiction.

Problem 7

Find an explicity conformal mapping from the upper half-plane slit along the vertical segment $\{iy \mid y > 0\} \setminus (0, 0 + ih]$ where $h > 0$ to the unit disc. Call this first domain Ω

Proof

Let's take a lot of small steps. First

$$\begin{aligned} f_1 &: z \mapsto z/(hi) \\ f_1 &: \Omega \to \Omega_1 := \{x + iy \mid x > 0\} \setminus \{x \mid x \geq 1\} \end{aligned}$$

then we square,

$$\begin{aligned} f_2 &: z \mapsto z^2 \\ f_2 &: \Omega_1 \to \Omega_2 := \mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty)) \end{aligned}$$

and then we connect these line segments by

$$\begin{aligned} f_3 &: z \mapsto 1/z - 1 \\ f_3 &: \Omega_2 \to \Omega_3 := \mathbb{C} \setminus (-\infty, 0] \end{aligned}$$

and take a square-root (negative reals branch cut)

$$\begin{aligned} f_4 &: z \mapsto \sqrt{z} \\ f_4 &: \Omega_3 \to \Omega_4 := \{x + iy \mid x > 0\} \end{aligned}$$

and finally conformally map the right half-plane to the disc

$$\begin{aligned} f_5 &: z \mapsto \frac{z-1}{z+1} \\ f_5 &: \Omega_4 \to \mathbb{D} \end{aligned}$$

Problem 8

Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Show that

$$|f(z)| \leq Ce^{a|z|}$$

for some constants C and a if and only if we have

$$\left|f^{(n)}(0)\right| \leq M^{n+1}$$

for some constant M.

Proof

First, suppose $|f(z)| \leq Ce^{a|z|}$ for all $z \in \mathbb{C}$. Applying Cauchy's integral formula (and the estimate that comes along with it), we get

$$\left|f^{(n)}(0)\right| \leq C\frac{n!}{r^n}e^{ar}$$

Choose $r = n/a$ and get

$$\begin{aligned}\left|f^{(n)}(0)\right| &\leq C\frac{n!}{r^n}e^{ar} \\ &\leq C\frac{n!a^n}{n^n}e^n \leq C(ea)^n\end{aligned}$$

Conversely, suppose $\left|f^{(n)}(0)\right| \leq M^{n+1}$ for each n and some constant M. Let's write the power series for f about 0:

$$\begin{aligned}f(z) &= \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(0)z^n \\ |f(z)| &\leq \sum_{n=0}^{\infty}\frac{1}{n!}\left|f^{(n)}(0)\right||z|^n \\ &\leq \sum_{n}\frac{1}{n!}M^{n+1}|z|^n = Me^{M|z|} < \infty\end{aligned}$$

so the series converges everywhere and we have the desired inequality globally.

Problem 9

Let $\Omega \subseteq \mathbb{C}$ be open and connected. suppose that (f_n) is a sequence of injective holomorphic functions defined on Ω such that $f_n \to f$ locally uniformly in Ω. Show that if f is not constant, then f is also injective in Ω.

Proof

We first prove a variant of Hurwitz's theorem. Recall that Hurwitz's theorem says that if $f_n \to f$ locally uniformly and all the f_n have no zeroes, then f is identically zero or has no zeroes.

We show the same result but about a single zero, i.e., if each f_n has at most one zero, then f does too (or else it's identically zero).

If f has no zeroes, we're done. Otherwise, f has at least one zero, w. If it's not isolated, $f \equiv 0$ and we're done. Thus around $\partial B(w, r)$, we have that f is never zero (and contains no other zeroes on the interior. Apply the argument principle with uniform convergence on $\partial B(w, r)$ and we're done.

Finally, note that if f is injective, $f - w$ has at most a single zero for every w. Since this is true for every f_n, it's true for f by the above lemma.

Problem 10

Let's introduce a vector space $\mathscr{B}$ defined as follows

$$\mathscr{B} := \left\{ u : \mathbb{C} \to \mathbb{C} \mid u \text{ is holomorphic and } \iint_{\mathbb{C}} |u(x+iy)|^2 e^{-(x^2+y^2)}\, dx\, dy < \infty \right\}$$

Show that $\mathscr{B}$ is a *complete* space when equipped with the norm

$$\|u\|^2 = \iint_{\mathbb{C}} |u(x+iy)|^2 e^{-(x^2+y^2)}\, dx\, dy$$

Proof

Suppose u_n is Cauchy in $\|\cdot\|$. Note that $ue^{-\frac{1}{2}(x^2+y^2)} \in L^2(\mathbb{C})$ is also Cauchy and so converges to some function $ue^{-\frac{1}{2}(x^2+y^2)}$. The goal is now to show that u is holomorphic. We'll show that $u_n \to u$ uniformly on compact sets.

Let $R > 0$. We'll show that $\{u_n\}$ restricted to $B(0, R)$ is Cauchy in L^∞. The

Cauchy integral formula rewritten (or rather the mean value theorem) gives us

$$\begin{aligned}
u_n(z) - u_m(z) &= \frac{1}{\pi R^2} \int_{B(z,R)} u_n(w) - u_m(w)\, dx\, dy \\
|u_n(z) - u_m(z)| &\leq \frac{1}{\pi R^2} \int_{B(z,R)} |u_n(w) - u_m(w)|\, dx\, dy \\
&\leq \frac{1}{\pi R^2} \left(\int_{B(z,R)} |u_n(w) - u_m(w)|^2\, dx\, dy \right)^{1/2} \left(\int_{B(z,R)} 1^2\, dx\, dy \right)^{1/2} \\
&\lesssim \frac{1}{\sqrt{\pi} R} \left(\int_{B(z,R)} |u_m(w) - u_m(w)|^2 e^{-(x^2+y^2)}\, dx\, dy \right)^{1/2} \\
&\leq \frac{1}{\sqrt{\pi} R} \|u_n - u_m\| \to 0
\end{aligned}$$

as long as z is selected from a compact set. Thus u is holomorphic, being the locally uniform limit of holomorphic functions.

Thus this space is complete.

Problem 11

Let $\Omega \subseteq \mathbb{C}$ be open, bounded, and simply connected. Let u be harmonic in Ω and assume that $u \geq 0$. Show the following: for each compact set $K \subseteq \Omega$, there exists a constant $C_K > 0$ such that

$$\sup_{x \in K} u(x) \leq C_K \inf_{x \in K} u(x)$$

Proof

This is the general Harnack inequality. Let's prove Harnack's inequality in a ball first, specically $B(0,R)$.

Let u be harmonic in $B(0,R)$ and consider the Poisson formula:

$$u(z) = \frac{1}{2\pi} \int_{\partial B(0,R)} \frac{R^2 - r^2}{R|z-w|^2} u(w)\, dw$$

where $|z| = r$.

We bound $R - r|z - w| < R + r$ and get

$$\begin{aligned}
\frac{R-r}{R+r} \fint u \leq u(z) \leq \frac{R+r}{R-r} \fint u \\
\frac{R-r}{R+r} u(0) \leq u(z) \leq \frac{R+r}{R-r} u(0)
\end{aligned}$$

Next, for the general Harnack inequality, take K compact and cover it with finitely many balls of radius $r/4$ where $r = \text{dist}(K, \Omega^c)$. Let $a = \sup_{x \in K} u(x)$ and $b = \inf_{x \in K} u(x)$. Since K is compact, these are both realized by $a = u(x)$ and $b = u(y)$.

Finally, using Harnack's inequality with inner radius $r/2$ and outer radius r, we compare a and b.

A different approach

By the Riemann mapping theorem, we can take $K \subseteq \Omega$ to $L \subseteq \mathbb{D}$ and apply Harnack's inequality there.

Problem 12

Let $\Omega = \{z \in \mathbb{C} \mid |z| > 1\}$. Suppose $u : \overline{\Omega} \to \mathbb{R}$ is bounded and continuous and subharmonic on the interior. Prove the following: if $u(z) \leq 0$ for all $|z| = 1$ then $u(z) \leq 0$ for all $z \in \Omega$.

Proof

A quick idea: consider $v(z) = u(1/z)$ which is subharmonic away from the origin (but set $v(0) = -\infty$ so that it's subharmonic everywhere) and then since v is subharmonic, it's bounded above by any harmonic function with the same boundary conditions. In particular, the 0 function is harmonic, so $v \leq 0$ so $u \leq 0$.

If this isn't legal, let's consider this instead: Let $v(z) = u(1/z)$ as before. Write $f(z) = v(z) - \varepsilon \log|1/z|$ which is still subharmonic on the punctured disc. Note that $f(z) \to -\infty$ as $z \to 0$. Thus eventually $f(z) \leq 0$ for $|z| \leq r$. Since $f(z)$ is subharmonic, $f(z) \leq 0$ on the annulus $r < |z| < 1$. This is by the maximum principle or subharmonicity and comparison to the constant function 0 (which is harmonic).

Then $f \leq 0$ everywhere because of how we selected r, and so sending $\varepsilon \to 0$, we're done.

7 S15

Problem 1

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)\,dx \right| = \int |f(x)|\,dx.$$

Proof

Note that for $f \geq 0$, we have

$$\sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f \right| = \int_{-n}^{n+1/n} f \to \int f = \int |f|$$

Let's show that for a simple function defined on closed intervals, the result holds. Set $V = \{c_i \chi_{[a_i,b_i]}\}$ where the closed intervals are disjoint.

If we pick n large enough that each $[k/n, (k+1)/n]$ intersects at most one of the intervals $[a_i, b_i]$, then the result trivially holds.

Furthermore, V is dense in $L^1(\mathbb{R})$. Suppose $\|f - s\|_{L^1} < \varepsilon$.

$$\begin{aligned}
\left| \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f \right| - \sum \left| \int_{k/n}^{(k+1)/n} s \right| \right| &\leq \sum_{k=-n^2}^{n^2} \left| \left| \int_{k/n}^{(k+1)/n} f \right| - \left| \int_{k/n}^{(k+1)/n} s \right| \right| \\
&\leq \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f - s \right| \leq \sum_{k=-n^2}^{n^2} \int_{k/n}^{(k+1)/n} |f - s| \\
&\leq \int_{-n}^{n+1/n} |f - s| < \varepsilon
\end{aligned}$$

This is the primary inequality needed. Everything else is bookkeeping.

Zach's Proof

Prove it for $f \geq 0$ and then trivially extend to all L^1 functions by considering $f = f^+ - f^-$.

Thanks Zachary Smith!

Problem 2

Let $f \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $g \in L^3_{\text{loc}}(\mathbb{R}^n)$. Assume that for all real $r \geq 1$, we have

$$\int_{r \leq |x| \leq 2r} |f(x)|^2 \, dx \leq r^a$$
$$\int_{r \leq |x| \leq 2r} |g(x)|^3 \, dx \leq r^b$$

where $a, b \in \mathbb{R}$ are such that $3a + 2b + n < 0$. Show that $fg \in L^1(\mathbb{R}^n)$.

Proof

We use polar coordinates and decompose $\{r > 0\}$ dyadically, ignoring $B(0,1)$, since there $g \in L^2_{\text{loc}}$ and thus $fg \in L^1_{\text{loc}}$ by Cauchy-Schwarz.

$$\begin{aligned}
\int_{\mathbb{R}^n} |fg| &= \sum_{N \in 2^{\mathbb{N}}} \int_{S^{n-1}} \int_N^{2N} |f(rv)g(rv)| r^{n-1} \, dv \, dr \\
&\leq \sum_{N \in 2^{\mathbb{N}}} \|f\|_{L^2(N<r<2N)} \|g\|_{L^3(N<r<2N)} \|1\|_{L^6(N<r<2N)} \\
&\lesssim \sum_{N \in 2^{\mathbb{N}}} N^{a/2} N^{b/3} N^{n/6} < \infty
\end{aligned}$$

Problem 3

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and let

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)| \, dy$$

be the Hardy-Littlewood maximal function.

1. Show that

   $$m\{Mf(x) > s\} \leq \frac{C_n}{s} \int_{|f(x)|>s/2} |f(x)| \, dx$$

   where $s > 0$ and the constant C_n depends on n only. The "Hardy-Littlewood maximal theorem" may be used.

2. Prove that if $\phi \in C^1(\mathbb{R})$, $\phi(0) = 0$, and $\phi' > 0$, then

   $$\int \phi(Mf(x)) \, dx \leq C_n \int |f(x)| \left(\int_{0<t<2|f(x)|} \frac{\phi'(t)}{t} \, dt \right) dx$$

Proof, Part 1

Recall the Hardy-Littlewood maximal theorem (the weak-type inequality):

$$m\{Mf > s\} \leq \frac{C_n}{s} \int |f|$$

Define $g = f\chi_{|f|>s/2}$.

Suppose $⨍_B |f| > s$. Then

$$\begin{aligned} ⨍_B |g| &= \frac{1}{m(B)} \int_{B\cap\{|f|>s/2\}} |f| + 0 \\ &= \frac{1}{m(B)} \int_B |f| - \frac{1}{m(B)} \int_{B\setminus\{|f|>s/2\}} |f| \\ &> s - s/2 = s/2 \end{aligned}$$

and so $Mf > s$ implies $Mg > s/2$. We thus get

$$\begin{aligned} m\{Mf > s\} &\leq m\{Mg > s/2\} \\ &\leq \frac{2C}{s} \|g\|_{L^1} \end{aligned}$$

Proof, Part 2

The trick here is to use the fundamental theorem of calculus.

$$\begin{aligned} \int \phi(Mf)\,dx &= \int \phi(Mf) - \phi(0) \\ &= \int_{\mathbb{R}^n} \int_0^{Mf(x)} \phi'(t)\,dt \\ &= \int_0^\infty \phi'(s) \int_{\{Mf>s\}} dx\,ds \\ &= \int_0^\infty \phi'(s)|Mf > s|\,ds \\ &\leq C_n \int_0^\infty \frac{\phi'(s)}{s} \int_{|f|>s/2} |f|\,dx \\ &= C_n \int |f(x)| \int_0^{2|f|} \frac{\phi'(s)}{s}\,ds\,dx \end{aligned}$$

It may be tempting instead to use the layer cake decomposition immediately to get

$$\int \phi(Mf) = \int_0^\infty \{\phi(Mf) > t\}\,dt$$

but we need ϕ^{-1} with this method and this proves not fruitful.

Problem 4

Let $f \in L^1_{\text{loc}}(\mathbb{R})$ be 2π-periodic. Show that linear combinations of translates $f(x-a)$ for $a \in \mathbb{R}$ are dense in $L^1((0, 2\pi))$ if and only if each Fourier coefficient of f is not zero.

Proof

Let $M = \overline{\text{span}\{f_a \coloneqq x \mapsto f(x-a)\}}$. Suppose $M \neq L^1$. Then there exists $g \in L^\infty$ such that $\int f_a g = 0$ for all a.

We calculate

$$\begin{aligned} 0 = \int f_a g &= \sum \hat{f}_a \hat{g} \\ &= \sum_{k \in \mathbb{Z}} e^{-2\pi i a k} \hat{f}(k) \hat{g}(k) \end{aligned}$$

which is a function h with $\hat{h}(k) = \hat{f}(k)\hat{g}(k)$. But since this is zero, it must have all its Fourier coefficients equal to zero by Fourier inversion. since $\hat{f}(k) \neq 0$, we must have $\hat{g} = 0$. Thus $g = 0$ and $M = L^1$.

Conversely, if $\hat{f}(k) \neq 0$, consider $\hat{g}(k) = 1$ and we'll have our evidence that $M \neq L^1$.

Problem 5

Let $u \in L^2(\mathbb{R})$ and set

$$U(x, \xi) = \int e^{-(x+i\xi-y)^2/2} u(y)\, dy.$$

Show that $U(x, \xi)$ is well-defined on $\mathbb{R}^2$ and that there exists a constant $C > 0$ such that for all $u \in L^2(\mathbb{R})$, we have

$$\iint |U(x, \xi)|^2 e^{-\xi^2}\, dx\, d\xi = C \int |u(y)|^2\, dy.$$

Determine C explicitly.

Proof

First, $|U(x,\xi)| \leq \|u\|_{L^2} \left\| e^{-(x+i\xi-y)^2/2} \right\|_{L^2(y)} \lesssim \left\| e^{-y^2} \right\|_{L^2} < \infty$, so this is well-defined.

To find C we expand $U(x,\xi)$:

$$U(x,\xi) = e^{-x^2/2} e^{\xi^2/2} e^{-ix\xi} \int e^{-y^2/2+xy} u(y) e^{i\xi y}\, dy$$

We group $f_x(y) = e^{-y^2/2+xy}u(y)$ and get

$$U(x,\xi) = e^{-x^2/2} e^{\xi^2/2} e^{-ix\xi} \hat{f}_x(-\xi).$$

Next, we write (noting that $\left|e^{-2ix\xi}\right| = 1$),

$$\begin{aligned}
\iint |U(x,\xi)|^2 e^{-\xi^2}\, dx\, d\xi &= \iint e^{-x^2+\xi^2} \left|\hat{f}_x(-\xi)\right|^2 e^{-\xi^2}\, dx\, d\xi \\
&= \int e^{-x^2} \int \left|\hat{f}_x(-\xi)\right|^2 d\xi\, dx \\
&= \int e^{-x^2} \int e^{-y^2} e^{2xy} |u(y)|^2\, dy\, dx \\
&= \int \int e^{-(x-y)^2} |u(y)|^2\, dy\, dx \\
&= \int e^{-z^2}\, dz \int |u(y)|^2\, dy
\end{aligned}$$

This yields $C = \sqrt{\pi}$.

Checking Adam Lott's solutions, it seems he used the convention of $e^{-2\pi ix\xi}$ for the Fourier transform, which introduces another 2π. I think there's probably a 2π hidden in the version of Plancherel I used, so we probably get $C = 2\pi\sqrt{\pi}$ instead.

Problem 6

Let B_1, B_2 be Banach spaces. We say that $T : B_1 \to B_2$ is compact if for any bounded sequence (x_n) in B_1, the sequence (Tx_n) has a convergence subsequence.

Show that if T is compact, then $\operatorname{im} T$ has a dense countable subset, i.e., $\operatorname{im} T$ is separable.

Proof

Suppose $\operatorname{im} T$ does not have a countable dense subset. In particular, consider $T(B(0,1)) = X$. We have that X is separable iff $\operatorname{im} T$ is (by taking countable scalings of our countable dense subset).

If X is not separable, it's not totally bounded, so for some $\varepsilon > 0$ we can find (Tx_i) with no convergent subsequence, despite the x_i being bounded by 1.

Problem 7

Suppose $f_n : \mathbb{D} \to \mathbb{C}^+ = \{\operatorname{Im} z > 0\}$ is a sequence of holmorphic functions and $f_n(0) \to 0$ as $n \to \infty$. Show that $f_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$.

Proof

Consider $g_n = 1/f_n$. These are holomorphic functions and by the stronger version of Montel's theorem are normal, as they all miss the same two points.

Thus for every subsequence there is a further subsequence which converges uniformly on compact sets to some function. By Hurwitz's theorem, the limit must have no poles unless it is ∞ constantly. But $g_n(0) \to \infty$, so $g_n \to \infty$ uniformly on compact sets (making the usual argument about every subsequence having a further subsequence).

Problem 8

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and suppose

$$\sup_{x\in\mathbb{R}}\{|f(x)|^2 + |f(ix)|^2\} < \infty \text{ and } |f(z)| \le e^{|z|}$$

for all $z \in \mathbb{C}$.

Deduce that f is constant.

Proof

Note that f is bounded on the real and imaginary axes. Let's show that f is bounded in the first quadrant, so from now on our domain is implicitly the first quadrant.

Let $\varepsilon > 0$ and define

$$h_\varepsilon(z) = e^{-\varepsilon\left(e^{-i\pi/4}z\right)^{3/2}}.$$

This is the Phragmen-Lindelof principle/method, like the Hadamard 3 Lines Lemma.

Note that the 3/2 power and the rotation guarantee that as $z \to \infty$, we have $h_\varepsilon(z) \to 0$ like $e^{-|z|^{3/2}}$. Thus $f(z)h_\varepsilon(z)$ is globally bounded in the first quadrant.

Since f is bounded by M on the real and imaginary axes and h_ε is at most 1, we have that fh_ε is bounded by M on some $B(0,R) \cap$ 1st quadrant. Then $|f| \leq M/|h_\varepsilon| \to M$ as $\varepsilon \to 0$. Thus f is bounded globally, so f is constant by Liouville's Theorem.

The general strategy

1. Find a collection of functions h_ε that are all bounded by 1 and go to zero quickly to infinity, but $h_\varepsilon \to 1$ pointwise or locally uniformly as $\varepsilon \to 0$.

2. Since h_ε is bounded by 1 on the boundary and goes to zero at ∞, it's bounded by 1 everywhere.

3. Pick a domain past which fh_ε is bounded by M (by decay at ∞)

4. On this domain, use $fh_\varepsilon \leq M$ on the boundary and maximum principle.

5. Therefore everywhere $fh_\varepsilon \leq M$, so send $\varepsilon \to 0$.

Problem 9

Let $\Omega = \{z \in \mathbb{C} \mid |z| > 1 \text{ and } \operatorname{Re} z > -2\}$. Suppose $u : \overline{\Omega} \to \mathbb{R}$ is bounded, continuous, and harmonic on Ω and that $u(z) = 1$ when $|z| = 1$ and $u(z) = 0$ when $\operatorname{Re}(z) = -2$.

Determine $u(2)$.

Proof

We need a Möbius transformation that turns this domain into an annulus. Try an automorphism of the disc:

$$\phi(z) = \frac{z-a}{1-\bar{a}z}$$

but naturally make a real. When we force $\phi(-2) = -x$ and $\phi(\infty) = x$ we get $a = \sqrt{3} - 2$ and $x = 1/(2-\sqrt{3})$.

Now we know that on the annulus with inner radius x and outer radius 1, there's only one harmonic function with boundary values 1 on the outer radius and 0 on the inner one: $f(z) = \frac{-1}{\log x}\log|z| + 1$.

Thus $u(2) = f(\phi(2))$. We get

$$\frac{-1}{\log x} \log\left|\frac{2-a}{1-2a}\right| + 1$$

Problem 10

Determine

$$\int_{-\infty}^{\infty} \frac{1}{(1+y^2)(1+(x-y)^2)}\, dy$$

for all $x \in \mathbb{R}$.

Proof

Take the contour that goes from $-R$ to R, then then back along a semicircle to $-R$. Let's call the semi-circle C_R and the line γ.

There are two poles in the upper half plane: $z = i$ and $(x - z) = -i$, so $z = x + i$.

Then

$$\int_{C_R} + \int_{\gamma} = 2\pi i \sum \operatorname{Res}.$$

There is a simple pole at i and a simple pole at $x + i$. After computation, sum is $2\pi/(x^2 + 4)$.

Problem 11

Let $\Omega = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Prove that for every bounded harmonic function $u : \Omega \to \mathbb{R}$ there is a harmonic function $v : \Omega \to \mathbb{R}$ satisfying

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof

In other words, every bounded harmonic function on the punctured disc, should be the real part of a holomorphic function on the punctured disc.

This becomes easier when we understand how harmonic functions on the disc look. First, I claim that

$$⨍_{C_r} u = \alpha \log r + \beta$$

for some α, β and where C_r is the circle of radius $r < 1$.

Furthermore, recall $\alpha = \int_{C_r} {}^*du$. Since $\alpha = 0$, $\int_{C_r} {}^*du = 0$.

Recall that for any cycle, $\int_\gamma du = 0$, and for any cycle homologous to zero, $\int_\gamma {}^*du = 0$.

Define $f = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ which is holomorphic. Then f has an anti-derivative F. Up to a constant then, $\operatorname{Re} F = u$. Write $\operatorname{Im} F = v$. Then u and v satisfy the Cauchy-Riemann equations.

Problem 12

Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ that obey

$$f'^2 + f^2 = 1$$

and prove your list is exhaustive.

Proof

All constants of magnitude 1 satisfy this.

Take a derivative of the equation. We get $2f'(f + f'') = 0$. Since zeroes are isolated, either $f' = 0$ identically (and we get the constants) or $f = -f''$ and $f = a\cos + b\sin$. The initial condition guarantees $a^2 + b^2 = 1$.

8 F15

Problem 1

Let g_n be a sequence of measurable functions on $\mathbb{R}^d$ such that $|g_n| \leq 1$ everywhere and assume $g_n \to 0$ ae. Let $f \in L^1$. Show that

$$f * g_n(x) = \int f(x-y) g_n(y)\, dy \to 0$$

uniformly on compact subsets of $\mathbb{R}^d$ as $n \to \infty$.

Proof

Note that on a compact subset K, $\|f * g_n\|_{L^\infty} \leq \|f\|_{L^1} \|g_n\|_{L^\infty} \lesssim 1$. Unfortunately, this doesn't give us convergence to zero.

Let $\varepsilon > 0$, $K \subseteq B(0,r) = B$ be compact, and take $A = B(0,R)$ so large that $\int_{B(0,R-r)^c} |f| < \varepsilon$. Next, take $E \subseteq A$ with measure at least $m(A) - \delta$ such that $g_n \to 0$ uniformly on E where δ is such that $\int_X |f| < \varepsilon$ whenever $m(X) < \delta$.

Finally,

$$\begin{aligned}
\|f * g_n\|_{L^\infty(B)} &= \operatorname*{ess\,sup}_{x\in B}\left|\int f(x-y)g_n(y)\,dy\right| \\
&\le \operatorname*{ess\,sup}_{x\in B}\int |f(x-y)||g_n(y)|\,dy \\
&\le \operatorname*{ess\,sup}_{x\in B}\int_A |f(x-y)||g_n(y)|\,dy + \operatorname*{ess\,sup}_{x\in B}\int_{A^c} |f(x-y)||g_n(y)|\,dy \\
&\le \operatorname*{ess\,sup}_{x\in B}\int_E |f(x-y)g_n(y)|\,dy + \operatorname*{ess\,sup}_{x\in B}\int_{A\setminus E} |f(x-y)g_n(y)|\,dy + \|f\|_{B(0,R-r)^c} \\
&\le \|f\|_{L^1}\|g_n\|_{L^\infty(E)} + \varepsilon + \varepsilon
\end{aligned}$$

Since the first term goes to 0 as $n \to \infty$, we're done.

Problem 2

Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, and let $a \in \mathbb{R}$ be such that $a > 1 - 1/p$. Show that the series

$$g(x) = \sum_{n=1}^{\infty}\int_n^{n+n^{-a}} |f(x+y)|\,dy$$

converges for almost all $x \in \mathbb{R}$.

Proof

Let's rewrite this.

$$\sum_{n=1}^{\infty} n^{-a}\int_0^1 |f(x+n+n^{-a}t)|\,dt$$

We can apply Hölder's inequality to get an L^p norm.

$$|g(x)| \le \left(\sum_{n=1}^{\infty} n^{-aq}\right)^{1/q}\left(\sum_{n=1}^{\infty}\left(\int_0^1 |f(x+n+n^{-a}t)|\,dt\right)^p\right)^{1/p}$$

We want an L^p norm instead of an L^1 norm. We can either try Minkowski's inequality, in which case we have an L^p norm of a nasty sum, or just use that

$[0, 1]$ is a finite measure space:

$$
\begin{aligned}
|g(x)| &\lesssim_{a,p} \left(\sum_{n=1}^{\infty} \left(\int_0^1 1^q \right)^{1/q} \int_0^1 |f(x+n+n^{-a}t)|^p \, dt \right)^{1/p} \\
&\lesssim \left(\sum_{n=1}^{\infty} \int_0^1 |f(x+n+n^{-a}t)|^p \, dt \right)^{1/p}
\end{aligned}
$$

To show that $g(x)$ is finite almost everywhere, we show it's L^p_{loc}. Let's integrate from N to $N+1$.

$$
\begin{aligned}
\int_N^{N+1} |g(x)|^p \, dx &\lesssim \int_N^{N+1} \sum_{n=1}^{\infty} \int_0^1 |f(x+n+n^{-a}t)|^p \, dt \, dx \\
&\le \int_0^1 \sum_{n=1}^{\infty} \int_N^{N+1} |f(x+n+n^{-a}t)|^p \, dx \, dt \\
&= \int_0^1 \sum_{n=1}^{\infty} \int_{N+n+n^{-a}t}^{N+1+n+n^{-a}t} |f(z)| \, dz \, dt \\
&\le \int_0^1 \sum_{n} \int_{N+n}^{N+n+2} |f(z)| \, dz \, dt \\
&\le \int_0^1 2\|f\|_{L^p} < \infty
\end{aligned}
$$

Problem 3

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ be such that for some $0 < p < 1$, we have

$$
\left| \int f(x)g(x) \, dx \right| \le \left(\int |g(x)|^p \, dx \right)^{1/p}
$$

for all $g \in C_0(\mathbb{R}^d)$. show that $f(x) = 0$ ae. Here $C_0(\mathbb{R}^d)$ is the space of continuous functions with compact support on $\mathbb{R}^d$.

Proof

Without the $1/p$, this would be saying that the linear functional $g \mapsto \int fg$ on L^p is zero (since the metric on L^p for $0 < p < 1$ is $\|\cdot\|_{L^p}^p$).

Instead, let K be compact. If we can show that $\left|\int_K f\right| \le (m(K))^{1/p}$, then by decomposing any set into small compact sets, the total integral will be 0.

Let $\varepsilon > 0$. Take $U \supseteq K$ with difference having measure less than ε. Take g continuous with $K \prec g \prec U$ continous.

Then

$$\left|\int_K f\right| \le \left|\int fg\right| + \varepsilon$$

and

$$\left|\int fg\right| \le (m(U))^{1/p} \le (m(K) + \varepsilon)^{1/p}$$

so sending $\varepsilon \to 0$ gives us the desired inequality.

Next, take any open set U of measure c and decompose it into approximately c/ε compact subsets E_i of small measure less than ε. Then

$$\begin{aligned}\left|\int_U f\right| \le \sum \left|\int_{E_i} f\right| &\le \sum m(E_i)^{1/p} \\ &\le \frac{c}{\varepsilon}\varepsilon^{1/p} = c\varepsilon^{1/p-1}\end{aligned}$$

which goes to 0 as $\varepsilon \to \infty$ since $1/p > 1$.

Problem 4

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and assume that (e_n) is an orthonormal system in $\mathcal{H}$. Let (f_n) be another orthonormal system which is complete (i.e., it's a Schauder basis, which means the closure of its span is $\mathcal{H}$).

1. Show that if $\sum_{n=1}^{\infty} \|f_n - e_n\|^2 < 1$ then (e_n) is also complete.
2. Assume that we only have $\sum_{n=1}^{\infty} \|f_n - e_n\|^2 < \infty$. Prove that (e_n) is still complete.

Proof, Part 1

To show (e_n) is complete, suppose $v \perp e_n$ for every n. Write $v = \sum \langle v, f_n \rangle f_n$. Furthermore, write $w = \sum \langle v, f_n \rangle e_n$. This is orthogonal to v.

We bound with Cauchy-Schwarz:

$$\|v - w\|^2 = \left\|\sum_{n=1}^{\infty} \langle v, f_n \rangle (f_n - e_n)\right\|^2 \le \left(\sum_{n=1}^{\infty} |\langle v, f_i \rangle|^2\right)\left(\sum_{n=1}^{\infty} \|f_n - e_n\|^2\right) \le \|v\|^2$$

but because w and v are orthogonal, $\|w\| = 0$ by the Pythagorean theorem, so $\langle v, f_n \rangle = 0$ for all n, so $v = 0$.

Proof, Part 2

Let's give up the first few f_n and e_n to be able to apply Part 1. Define $E_n = \overline{\operatorname{span}\{e_n, e_{n+1}, \dots\}}$ and $F_n = \overline{\operatorname{span}\{f_n, f_{n+1}, \dots\}}$.

Note that eventually e_n and f_n are close, so E_n and F_n should also be "close" when n is large.

Claim: $\left\|\pi_{E_n} - \pi_{F_n}\right\| \to 0$ in operator norm as $n \to \infty$. Let $\|x\| = 1$.

$$\begin{aligned}
\left\|\pi_{E_n}(x) - \pi_{F_n}(x)\right\| &= \left\|\sum_{k\geq n} \langle x, e_k\rangle e_k - \langle x, f_k\rangle f_k\right\| \\
&\leq \sum_{k\geq n} \|\langle x, e_k\rangle(e_k - f_k) + \langle x, e_k\rangle f_k - \langle x, f_k\rangle f_k\| \\
&\leq \sum_{k\geq n} |\langle x, e_k\rangle| \|e_k - f_k\| + \sum_{k\geq n} \langle x, e_k - f_k\rangle \|f_k\| \\
&\leq \sum_{k\geq n} \|e_k - f_k\| + \sum_{k\geq n} \|e_k - f_k\| \to 0
\end{aligned}$$

Note that $\left\|\pi_{E_n^\perp} - \pi_{F_n^\perp}\right\| = \left\|\pi_{E_n} - \pi_{F_n}\right\|$.

We want to show that $\{e_1, \dots, e_{n-1}\}$ span $E_n^\perp$. We know they're independent, so their span has dimension $n-1$. We just need to check that $E_n^\perp$ has the right dimension. We know that $F_n^\perp$ does.

Take n large enough that $\left\|\pi_{E_n^\perp} - \pi^\perp_{F_n}\right\| < 1/2$. Claim: $\dim E_n^\perp = \dim F_n^\perp$. Written abstractly, if $\|\pi_A - \pi_B\| < 1/2$ and $\dim B = k < \infty$, then $\dim A = k$.

We actually just need $\dim A \leq k$.

Let $x_1, \dots, x_{k+1}$ be $k+1$ many vectors in A. Then

$$0 = \alpha_1 \pi_B(x_1) + \cdots + \alpha_{k+1}\pi_B(x_{k+1})$$

and so

$$\begin{aligned}
\|\alpha_1 x_1 + \cdots + \alpha_{k+1}x_{k+1}\| &= \|\pi_A(\dots) - \pi_B(\dots)\| \\
&\leq \|\pi_A(\dots)\| \leq \frac{1}{2}\|\dots\|
\end{aligned}$$

and so $\alpha_1 x_1 + \cdots = 0$ so we have linear dependence. This concludes the proof.

Problem 5

Show that the Holder continuous functions are meager in $C([0,1])$.

Proof

Note that if f is \$a\$-Hölder then it is \$b\$-Hölder also if $b < a$. Furthermore, if f is \$a > 1\$-Hölder, then it is constant (since it is differentiable with derivative equal to zero).

Let $E_n = \{f \in C([0,1]) \mid f \text{ is } 1/n \text{ Hölder cont}\}$. Note that $E_n \subseteq E_m$ when $n < m$.

It's enough to show that E_n is meager for each n.

Fix $\delta = 1/n$. Now write $E = E_n = \bigcup_N \{f \mid |f(x) - f(y)| \le N\|x - y\|^\delta\} = \bigcup A_N$.

If we can show that each A_N is meager, we're done. This is easy by taking an appropriate "spike" function that looks like x^a for some appropriate a. Details can be found in Adam Lott's solutions.

Problem 6

Let $u \in L^2(\mathbb{R}^d)$ and let us say that $u \in H^{1/2}(\mathbb{R}^d)$ (a Sobolev space) if $\langle \xi \rangle^{1/2} \hat{u}(\xi) \in L^2(\mathbb{R}^d)$.

Show that $u \in H^{1/2}$ iff

$$\iint \frac{|u(x+y) - u(x)|^2}{|y|^{d+1}}\, dx\, dy < \infty$$

Proof

Let's expand the double integral:

$$\begin{aligned}\int \frac{1}{|y|^{d+1}} \int |u(x+y) - u(x)|^2\, dx\, dy &= \int \frac{1}{|y|^{d+1}} \int \left|1 - e^{2\pi i y \cdot \xi}\right| |\hat{u}(\xi)|^2\, d\xi\, dy \\ &= \int |\hat{u}(\xi)|^2 \int \frac{|1 - e^{2\pi i y \cdot \xi}|}{|y|^{d+1}}\, dy\, d\xi\end{aligned}$$

Thus our goal is to show that $\int \frac{|1-e^{2\pi i y \cdot \xi}|}{|y|^{d+1}}\, dy$ is comparable to $|\xi|$ so that we get $\int |\hat{u}(\xi)|^2 |\xi|\, d\xi$.

First, an upper bound on the integral. When y is large, just bound the numerator by 4. When y is small (specifically, when $2\pi i y \cdot \xi$ is small), the numerator is smaller than the exponent (one can bound $|1 - e^z| \le 2|z|$ for $|z| <$

1/2).

$$\begin{aligned}\int \frac{|1-e^{2\pi i y\cdot\xi}|}{|y|^{d+1}}\,dy &= \int_{|y|<1/2|\xi|} + \int_{|y|>1/2|\xi|} \\ &\le \int_{|y|<1/2|\xi|} \frac{|4\pi y\cdot\xi|^2}{|y|^{d+1}}\,dy + \int_{|y|>1/2|\xi|} \frac{4}{|y|^{d+1}}\,dy \\ &\lesssim |\xi|\end{aligned}$$

We basically just take the first term and restrict to when $|y\cdot\xi|$ is large. Integrate y over $\{|y| \le 1/(3|\xi|)$ and $|y\cdot\xi| \ge (1/2)|y||\xi|$. Use the same inequality.

Since the second condition depends only on the direction of y and not the magnitude, we can make the constant uniform in ξ.

Problem 7

Assume that f is analytic in $\mathbb{D}$ and continuous on the boundary. If $f(z) = f(1/z)$ on the boundary, prove that f is constant.

Proof

Extend f to the Riemann sphere by setting $f(z) = f(1/z)$ when $|z| > 1$. The extension is continuous on the boundary of the circle, and so f is analytic on the entire Riemann sphere with no poles. This folows by Morera's theorem: take any curve around a point on the circle and split it into two curves contained entirely in the disc and entirely outside of the closure of the disc. Integrals on the matching pieces annihilate in the limit as the two blobs get closer.

This is poorly described here, but you can figure it out.

Clearly, f is a constant by Liouville.

Problem 8

Assume that $f(z)$ is an entire function that is 2π-periodic (in the sense that $f(z+2\pi) = f(z)$ and

$$|f(x+iy)| \le Ce^{\alpha|y|}$$

for some $C > 0$ where $0 < \alpha < 1$. Prove that f is constant.

Proof, Method 1 (A clever trick with pullbacks)

Note that we can write $f(z) = g(e^{iz})$ by the Monodromy theorem. Alternatively, $g(z) = f(\log z/i)$. Note that

$$|g(z)| \le Ce^{\alpha \operatorname{Im}(\log z/i)} \le Ce^{\alpha |\log z|}$$

Then sending $z \to 0$ we get eventually $|g(z)| \le |z|^{-\alpha}$ (because $\log \to -\infty$ so the absolute value introduces a sign).

Since $\alpha < 1$, $g(z)z$ goes to zero, so g has a removable singularity.

This means that f is bounded as $\operatorname{Im}(z) \to \infty$.

Similarly, by taking a simple rotation first, f is bounded as $\operatorname{Im}(z) \to -\infty$.

Thus f is constant by Liouville's theorem.

Wrong Non-Proof (Inspired by Hadamard's Three Line Lemma and the Phragmen-Lindelof method)

This doesn't work.

If we had that f was bounded on the boundary of the strip, we could run the following argument, but we can't:

It's enough to show that f is bounded in $[0, 2\pi] \times \mathbb{R}$. Consider

$$g_n(z) = e^{-(z^2+1)/n}$$

This is holomorphic, lacks zeroes, and if we consider f/g_n. Note that f/g_n is bounded in the strip globally, since as $y \to \infty$, $|g_n(x + iy)| \lesssim e^{-|y|^2/n} \to 0$.

Also note that $|g_n| \ge 1$, so $1/g_n$ is bounded by 1.

If f were bounded on the boundary of the strip by K, find where f/g_n is also bounded by K and send $n \to \infty$ after deducing that f/g_n is bounded by K on the interior. Then send the domain to infinity (or do this before $n \to \infty$). This is the Phragmen-Lindelof method.

Problem 9

Let (f_j) be a sequence of entire functions that, writing $z = x + iy$, we have

$$\iint_{\mathbb{C}} \left|f_j(z)\right|^2 e^{-|z|^2}\, dx\, dy \le C$$

for some constant $C > 0$. Show that there exists a subsequence (f_{j_k}) and an entire function f such that

$$\iint_{\mathbb{C}} \left|f_{j_k}(z) - f(z)\right|^2 e^{-2|z|^2}\, dx\, dy \to 0$$

as $k \to \infty$.

Proof

Let's show we have a normal family. It suffices to prove that the f_j are locally uniformly bounded.

We use the usual trick of comparing L^2 norms with L^∞ norms via the mean value property. Take $K = \overline{B(w,1)}$. We write

$$\begin{aligned} C \geq \iint_K \left|f_j(z)\right|^2 e^{-|z|^2}\,dx\,dy &\geq \min_K e^{-|z|^2} \iint_K \left|f_j(z)\right|^2 \\ &\gtrsim \left(\iint_K \left|f_j(z)\right|\right)^2 \\ &\geq \left|f_j(w)\right|^2 \end{aligned}$$

thus the f_j are a normal family. Suppose $f_{j_k} \to f$ locally uniformly.

By considering constants, it turns out that $\left|f_j(z)\right| \lesssim e^{(|z|+1)^2/2}$.

Suppose $\iint_{\mathbb{C}} \left|f_{j_k}(z) - f(z)\right|^2 e^{-2|z|^2}\,dx\,dy \not\to 0$. Then there's a subsequence that stays bounded below by $\varepsilon > 0$.

If this sequence stays bounded, then by the same argument, $f_{j_{k_\ell}} - f$ is a normal family, but it has to converge to 0, a contradiction. But we can't guarantee these integrals stay bounded. Instead,

$$\begin{aligned} \iint_{\mathbb{C}} \left|f_{j_k}(z) - f(z)\right|^2 e^{-2|z|^2}\,dx\,dy &\leq \int_{B(0,r)} \cdots + \int_{|z|>r} \left|f_{j_k}(z) - f(z)\right|^2 e^{-2|z|^2}\,dx\,dy \\ &\lesssim o(1) + \int_{|z|>r} e^{|z|^2+2|z|+1} e^{-2|z|^2}\,dx\,dy \\ &\lesssim o(1) + \int_{|z|>r} e^{-|z|^2+2|z|+1}\,dx\,dy \end{aligned}$$

Send $r \to \infty$ to bound the integral.

Problem 10

Use the Residue Theorem to prove that

$$\int_0^\infty e^{\cos x} \sin(\sin x) \frac{1}{x}\,dx = \frac{\pi(e-1)}{2}$$

Proof

This is odd, so we want

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{\cos x}\sin(\sin x)\frac{1}{x}\,dx$$

Unfortunately, this has no poles if we extend it to $\mathbb{C}$ in the obvious way. Instead, we recognize this integrand as $\operatorname{Im}(e^{e^{ix}})/x$ which can extend in a way that has a pole. The pole at 0 has residue e.

We consider a small clockwise semicircle around the origin from $-r$ to r called γ_r, a line segment to R, then counter-clockwise to $-R$, then a line segment to $-r$.

First, note that

$$\int_{\gamma_r} e^{e^{iz}}\frac{1}{z}\,dz \to -\frac{1}{2}(2\pi i)e = -\pi i e$$

Next on γ_R, we write $z = Re^{i\theta}$. Then

$$\begin{aligned}\left|\int_0^{\pi} \exp\left(e^{iRe^{i\theta}}\right)\frac{1}{Re^{i\theta}}iRe^{i\theta}\,d\theta\right| &\leq \int_0^{\pi}\left|\exp\left(e^{iRe^{i\theta}}\right)\right|d\theta \\ &\leq \int_0^{\pi}\exp\left|e^{-R\sin\theta}\right| \leq \int e < \infty\end{aligned}$$

so we can use Dominated Convergence to get that $\int_{\gamma_R} f \to i\pi$.

Finally, we join all the curves and note that no poles are contained within to get

$$0 = 2\int_r^R f(z)\,dz + \int_{\gamma_r} + \int_{\gamma_R}$$

so sending $r \to 0$ and $R \to \infty$ gives us

$$\int_0^{\infty} f(z)\,dz = \frac{1}{2}(\pi i e - \pi i)$$

so the imaginary part is $\frac{\pi}{2}(e-1)$ as desired.

Problem 11

Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ and let u be harmonic in Ω and continuous in the closure such that

$$u(x, y) \leq |x + iy|$$

for large $(x, y) \in \Omega$. Assume that

$$u(x, 0) \leq ax$$
$$u(0, y) \leq bx$$

for $x, y \geq 0$ and some $a, b > 0$. Show that

$$u(x, y) \leq ax + by$$

in Ω.

Proof

Let's use the Phragmen-Lindelof principle. We need an auxiliary function which decays, but not too fast, with a parameter that we can send to 0 that makes the function go to 1.

Consider

$$g(z) = \varepsilon r^{3/2} \cos\left(\frac{-3\pi}{8} + \frac{3\theta}{2}\right)$$

which is the real part of a holomorphic function and hence harmonic.

Write $\phi(z) = ax + by + g(z)$ and $v = u - \phi$.

Note that ϕ is harmonic, so v is subharmonic still and has no local maximum in the interior of the domain Ω.

Note that for large r, $v(z) \leq r - \varepsilon r^{3/2} \to -\infty$, so we can pick R such that $v \leq 0$ on the circle of radius R intersected with the first quadrant.

On the x-axis and y-axis, $v \leq 0$, so it must be true globally. Increasing $R \to \infty$ gives us this really globally. Send $\varepsilon \to 0$ and get that $u \leq ax + by$ globally.

Problem 12

Find a function $u(x, y)$ harmonic on the region between the circles $|z| = 2$ and $|z - 1| = 1$ which equals 1 on the outer circle and 0 on the inner circle (except at the points where the circles are tangent to each other).

Proof

The trick is to send the intersection point to ∞ via a Möbius transformation.

This sends the circle $|z-1| = 1$ to $\operatorname{Re}(z) = -1/2$ and the circle $|z| = 2$ to $\operatorname{Re}(z) = -1/4$. Thus we should take

$$\operatorname{Re}\left(\frac{4}{z-2} + 2\right)$$

as our function.

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Problem 1

Let $K_t(x) = (4\pi t)^{-3/2} e^{-|x|^2/4t}$ where $x \in \mathbb{R}^3$, $t > 0$.

1. Show that the linear map

   $$L^3(\mathbb{R}^3) \ni f \mapsto t^{1/2} K_t * f \in L^\infty(\mathbb{R}^3)$$

   is bounded uniformly in $t > 0$. Here,

   $$K_t * f(x) = \int_{\mathbb{R}^3} K_t(x-y) f(y)\, dy$$

   is the convolution.

2. Prove that $t^{1/2} \|K_t * f\|_{L^\infty} \to 0$ as $t \to 0$ for $f \in L^3(\mathbb{R}^3)$.

Proof, Part 1

We apply Young's inequality and the known Gaussian integral:

$$\begin{aligned}
t^{1/2}\|K_t * f\|_{L^\infty} &\le t^{1/2}\|f\|_{L^3}\|K_t\|_{L^{3/2}} \\
&\le \|f\|_{L^3}\frac{1}{(4\pi)^{3/2}t}\left(\int_{\mathbb{R}^3}\left(e^{-|x|^2/4t}\right)^{3/2}\,dx\right)^{2/3} \\
&\le \|f\|_{L^3}\frac{1}{(4\pi)^{3/2}t}\left(\prod_{i=1}^{3}\int_{-\infty}^{\infty} e^{-3x_i^2/8t}\,dx_i\right)^{2/3} \\
&= \|f\|_{L^3}\frac{1}{(4\pi)^{3/2}t}\left(\int_{-\infty}^{\infty} e^{-3x^2/8t}\,dx\right)^{2} \\
&\le \|f\|_{L^3}\frac{1}{(4\pi)^{3/2}t}\left(\int_{-\infty}^{\infty} e^{-3x^2/8t}\,dx\right)^{2} \\
&= \|f\|_{L^3}\frac{1}{(4\pi)^{3/2}t}\frac{8\pi t}{3} \\
&= \|f\|_{L^3}\frac{1}{3\sqrt{\pi}}
\end{aligned}$$

Proof, Part 2

Suppose $g \in C_c^\infty$. Then

$$\|K_t * g\|_{L^\infty} \le \|K_t\|_{L^1}\|g\|_{L^\infty} \lesssim \|g\|_{L^\infty}$$

since K_t is integrable with bounded L^1 norm independent of t.

Thus $t^{1/2}\|K_t * g\|_{L^\infty} \lesssim t^{1/2} \to 0$ as $t \to 0$.

Next, let $\varepsilon > 0$, and take $g \in C_c^\infty$ with $\|g - f\|_{L^3} < \varepsilon$.

Then

$$\begin{aligned}
t^{1/2}\|K_t * f\|_{L^\infty} &\le t^{1/2}\|K_t * (f-g)\|_{L^\infty} + t^{1/2}\|K_t * g\|_{L^\infty} \\
&\le \frac{1}{3\sqrt{\pi}}\|f-g\|_{L^3} + \varepsilon
\end{aligned}$$

as $t \to 0$. Thus the whole thing goes to zero as $t \to \infty$.

Problem 2

Let $f \in L^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}f(x-\sqrt{n})$$

converges absolutely for almost all $x \in \mathbb{R}$.

Proof

We show that this quantity is locally integrable. Let $M \in \mathbb{Z}$.

$$\begin{aligned}
\int_M^{M+1} \left| \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} f(x-\sqrt{n}) \right| dx &\le \int_M^{M+1} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left| f(x-\sqrt{n}) \right| dx \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_M^{M+1} \left| f(x-\sqrt{n}) \right| dx \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_{M-\sqrt{n}}^{M+1-\sqrt{n}} |f|\, dx \\
&= \sum_{n=1}^{\infty} \sum_{k=n^2}^{(n+1)^2-1} \frac{1}{\sqrt{k}} \int_{M-\sqrt{k}}^{M+1-\sqrt{k}} |f|\, dx \\
&\le \sum_{n=1}^{\infty} \sum_{k=n^2}^{(n+1)^2-1} \frac{1}{n} \int_{M-n-1}^{M-n+1} |f|\, dx \\
&\le \sum_{n=1}^{\infty} 3 \int_{M-n-1}^{M-n+1} |f|\, dx \le 6\|f\|_{L^1}
\end{aligned}$$

Problem 3

Let $f \in L^1_{\text{loc}}(\mathbb{R})$ be real-valued and assume that for each integer $n > 0$, we have

$$f\left(x + \frac{1}{n}\right) \ge f(x)$$

for almost all $x \in \mathbb{R}$. Show that for each real number $a \ge 0$, we have

$$f(x+a) \ge f(x)$$

for almost all $x \in \mathbb{R}$.

Proof

Notice that for almost every x, $f(x+1/n) \ge f(x)$ holds for every $\mathbb{N} \ni n > 0$. Even stronger, for almost every x, we have that $f(x+q) \ge f(x)$ for all $q \in \mathbb{Q}$ with $q > 0$.

Let $\varepsilon > 0$. Let x be a Lebesgue point satisfying the above conditions.

Fix $a \in \mathbb{R}$ and $r < a/2$ small enough that the following two conditions holds:

$$\fint_{x+a-r}^{x+a+r} |f(y) - f(x+a)|\, dy < \varepsilon/4$$

$$\fint_{x-r}^{x+r} |f(y) - f(x)|\, dy < \varepsilon/2$$

Fix $a/2 > s > 0$. Then for $r/2 < s < r$, we have that $(x-s, x+s) + q \subseteq (x+a-r, x+a+r)$ for some positive rational. Thus

$$\begin{aligned}
|f(x) - f(x+a)| &\leq \left|\fint_{x-s}^{x+s} f(y) - f(x)\, dy\right| + \left|\fint_{x-s}^{x+s} f(y) - f(x+a)\, dy\right| \\
&\leq \varepsilon/2 + \frac{1}{2s}\int_{x-s}^{x+s} |f(y+q) - f(x+a)|\, dy \\
&\leq \varepsilon/2 + 2\int_{x+a-r}^{x+a+r} |f(y) - f(x+a)|\, dy \\
&< \varepsilon
\end{aligned}$$

where the second bound holds for almost every y in the integrand.

Alternate Proof

(See Adam Lott's solutions). The gist of this alternate solution is that we consider $\int_b^c f(y+a) \geq \int_b^c f(y)$ and write a in binary and argue by convergence of the series for a with Lebesgue Dominated Convergence.

Problem 4

Let V_1 be a finite dimensional subspace of the Banach space V. Show that there is a continuous projection $P : V \to V_1$, i.e., a continuous linear map $P : V \to V$ such that $P^2 = P$ and the image of P is equal to V_1.

Proof

This is a standard result.

Write $V_1 = \text{span}\{x_1, \ldots, x_n\}$. For each x_i, consider the following functional $p_i : V_1 \to V_1$ via $p_i(a_1x_1 + \cdots + a_nx_n) = a_i$.

This is obviously continuous on V_1, and so can be extended by Hahn-Banach to a continuous functional P_i.

Set $P = P_1x_1 + \cdots + P_nx_n$. This is continuous and has image V_1 where it's the identity.

Problem 5

For $f \in C_0^\infty(\mathbb{R}^2)$, define $u(x,t)$ via

$$u(x,t) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} \frac{\sin(t|\xi|)}{|\xi|} f(\xi)\, d\xi$$

for $x \in \mathbb{R}^2$ and $t > 0$.

Show that $\lim_{t\to\infty} \|u(\cdot,t)\|_{L^2} = \infty$ for a set of f that is dense in $L^2(\mathbb{R}^2)$.

Proof

Write $u(x,t) = \mathcal{F}^{-1}(g_t(\xi))$ where $g_t(\xi) = \sin(t|\xi|)f(\xi)/|\xi|$.

We begin with Plancherel:

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2} &= \|g_t\|_{L^2} \\
&= \int_{\mathbb{R}^2} t^2 \left(\frac{\sin(t|\xi|)}{t|\xi|}\right)^2 |f(\xi)|^2\, d\xi \\
&\geq \int_{\mathbb{R}^2} \left(\frac{\sin(|\zeta|)}{|\zeta|}\right)^2 |f(\zeta/t)|^2\, d\zeta \\
&\geq C \inf_{\zeta\in B(0,1)} |f(\zeta/t)|^2 \\
&\geq C \inf_{\xi\in B(0,1/t)} |f(\xi)|^2
\end{aligned}$$

We would like this to go to ∞ as $t \to \infty$. Thus we consider $S = \{f \in L^2 \mid \lim_{x\to 0} |f(x)| = \infty\}$.

Then for each $f \in S$, the desired result holds.

Finally, let's show that S is dense. Given any $f \in L^2 \cap C_c^\infty$, we simply add $\varepsilon|x|^{-1/2}\chi_{B(0,1)}$.

Problem 6

Suppose that $\{\phi_n\}$ is an orthonormal system of continuous functions in $L^2([0,1])$ and let S the the closure of the span of $\{\phi_n\}$. If

$$\sup_{f\in S\setminus\{0\}} \frac{\|f\|_{L^\infty}}{\|f\|_{L^2}} < \infty$$

prove that S is finite dimensional.

Proof

Define $E_x : S \to \mathbb{R}$ via $E_x(f) = f(x)$. Note that $|E_x(f)| \le \|f\|_{L^\infty} \le M\|f\|_{L^2}$, so E_x is a bounded linear functional on $L^2 \cap S$.

Note that S is closed and thus a Hilbert space in its own right, so by the Riesz representation theorem, there is $g_x \in S$ such that

$$\langle f, g_x \rangle = f(x)$$

for all $f \in S$.

Moreover, notice that

$$\|g_x\|^2 = |g_x(x)| \le \|g_x\|_{L^\infty} \le M\|g_x\|_{L^2}$$

so $\|g_x\|_{L^2} \le M$ for every $x \in [0, 1]$.

Finally,

$$M^2 \ge \|g_x\|_{L^2}^2 \qquad \ge \sum_n |\langle \phi_n, g_x \rangle|^2 = \sum |\phi_n(x)|^2$$

by Bessel's inequality.

Integrating both sides, we get $\#\{\phi_n\} \le M^2 < \infty$.

Problem 7

Determine

$$\int_0^\infty \frac{x^{a-1}}{x+z}\, dx$$

for $0 < a < 1$ and $\operatorname{Re} z > 0$.

Proof

Let's try a "Pac-Man" contour. We cut out the positive real axis as our branch cut for log. We then integrate on a large circle of radius R, then along the lower side of the positive reals from R to ε, then backwards along a tiny circle of radius ε, then along the top side of the positive reals up to R. Call the small circular arc C_ε, the large one C_R, and the lines L_+ and L_-.

The straight line integrals are

$$\int_{L^+} \frac{\exp((a-1)(\log x + \theta i))}{x+z} + \int_{L^-} \frac{\exp((a-1)(\log x + (2\pi - \theta)i))}{x+z}$$
$$\to \left(1 + \exp(2\pi i(a-1))\right) \int_{\mathbb{R}} \frac{x^{a-1}}{x+z}\,dx$$
$$= \left(1 - \exp(2\pi i a)\right) \int_{\mathbb{R}} \frac{x^{a-1}}{x+z}\,dx$$

where θ is the small angle these lines are placed at.

The integral on the large circle is bounded by $2\pi R \frac{R^{a-1}}{R} \to 0$ since $a - 1 < 0$. The integral on the small circle easily also goes to 0.

We now check for the residues in the slit plane.

Since $\mathrm{Re}(z) > 0$, $-z$ is not on the positive real axis, and there is exactly one pole in the slit plane with residue $\exp((a-1)\log(-z))$. Thus,

$$\left(1 - \exp(2\pi i a)\right) \int_{\mathbb{R}} \frac{x^{a-1}}{x+z}\,dx = 2\pi i \exp((a-1)\log(-z))$$
$$\int_{\mathbb{R}} \frac{x^{a-1}}{x+z}\,dx = \frac{2\pi i(-z)^{a-1}}{1 - \exp(2\pi i a)}$$

Problem 8

Let $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ and let $f_n : \mathbb{C}_+ \to \mathbb{C}_+$ be a sequence of holomorphic functions. Show that unless $|f_n| \to \infty$ uniformly on compact subsets of $\mathbb{C}_+$, there exists a subsequence converging uniformly on compact subsets of $\mathbb{C}_+$ (presumably to something holomorphic).

Proof

Hurwitz's theorem guarantees that given a subsequence of f_n converging locally uniformly to some meromorphic function g, g may have no poles.

Since $1/f_n$ has no zeroes and is holomorphic, we have that either $|1/f_n| \to 0$ uniformly on compact sets and hence $g \equiv \infty$, or g has no poles.

Normality in the *classical sense* is that these holomorphic functions have some subsequence which converges to a continuous (think meromorphic) function with repsect to the spherical metric. However, there's one possible continuous function which is not meromorphic: the constant ∞ function. This means normal in the classical sense is equivalent to:

1. Either $|f_n| \to \infty$ uniformly on compact subsets, or

2. there exists a subsequence convering uniformly on compact subsets of $\mathbb{C}_+$ (to something holomorphic by Hurwitz).

Therefore it suffices to prove normality. We have a few options here.

- Method 1

  First, recall that normality holds if and only if the spherical derivatives are locally bounded, the spherical derivatives being

$$\rho(f) = \frac{2|f'|}{1 + |f|^2}$$

  Consider the map $\mathbb{C}_+ \to \mathbb{D}$ via $\phi(z) = \frac{z-i}{z+i}$. Note that $\phi \circ f_n$ is normal (in the usual sense, and hence in the classical sense, since these functions all avoid infinity), being holomorphic and uniformly bounded. Thus, calling $g_n = \phi \circ f_n$,

$$\begin{aligned}\frac{2|g_n'|}{1 + |g_n|^2} &= \frac{8\frac{|f_n'|^2}{|f_n+i|^2}}{1 + \frac{|f_n-i|^2}{|f_n+i|^2}} \\ &= \frac{8|f_n'|^2}{|f_n + i|^2 + |f_n - i|^2} \\ &= 2\frac{2|f_n'|^2}{1 + |f_n|^2}\end{aligned}$$

  Here, Adam Lott's solutions write $|f_n'|$ in the numerator instead and identify $\rho(f_n)$ as also being uniformly bounded. I don't see how to get rid of the extra $|f_n'|$ though.

- Method 2

  Recall Montel's theorem. A family of holomorphic functions that omit two shared points is normal. Here normal means normal with respect to the codomain $\mathbb{C}^*$, but Hurwitz's theorem implies that sequences of holomorphic functions that converge locally uniformly satisfy that the limit function is either constantly ∞ or lacks poles (consider $1/f$).

  We're done.

Problem 9

Let $f : \mathbb{C} \to \mathbb{C}$ be entire and assume $|f(z)| = 1$ when $|z| = 1$. Show that $f(z) = Cz^m$ for some integer $m \geq 0$ and $C \in \mathbb{C}$ with $|C| = 1$.

Proof

First, consider the Blaschke product $B(z)$ which shares zeroes with f on $\mathbb{D}$. Then f/B has no zeroes and magnitude 1 on the disc. The same is true of B/f. In particular, $|f| = |B|$ inside the disc. Thus $f = CB$ for some constant C of magnitude 1, since they're both analytic.

Note that Blaschke products all have poles unless their zeroes are all at the origin. Thus $f = Cz^m$ for some m.

Problem 10

Does there exist a holomorphic function $f(z)$ in the disc $\mathbb{D}$ such that $\lim_{|z|\to 1}|f(z)| = \infty$. Either find one or prove that none exist.

Proof

No! Note that f has finitely many zeros, and so after dividing by a Blaschke product, f is zero-less in the disc. Let's take $1/f$. This is zero on the boundary of the disc, and achieves a maxmimum in the interior, and is holomorphic. The maximum princple guarantees that this is impossible.

Problem 11

Assume that $f(z)$ is holomorphic on $|z| < 2$. Show that

$$\max_{|z|=1}\left|f(z) - \frac{1}{z}\right| \geq 1$$

Proof

Note that if C is the circle of radius 1 at the origin,

$$\int_C f(z) - 1/z\, dz = -2\pi i$$
$$2\pi = \left|\int_C f(z) - 1/z\right| \leq 2\pi \max_{|z|=1}|f(z) - 1/z|$$

and we're done.

Alternate Proof

Suppose instead $|f(z) - 1/z| < 1 = |1/z|$. Fixing this, $|zf(z) - 1| < |z| = 1$ on C, so $zf(z)$ has no zeroes inside of $\mathbb{D}$, which means f has a pole. (This is by considering the image of C under $zf(z)$ and the argument principle.) This is impossible.

Alternatively, Rouche's theorem says that $|zf(z) - 1| < 1$ implies $zf(z)$ has the same number of zeroes in $\mathbb{D}$ as the constant function 1. This is crazy.

Problem 12

1. Find a real valued harmonic function v defined on the disc $|z| < 1$ such that $v(z) > 0$ and $\lim_{z \to 1} v(z) = \infty$.

2. Let u be a real-valued harmonic function in the disc such that $u(z) \le M < \infty$ and $\limsup_{r \to 1} u(re^{i\theta}) \le 0$ for all $\theta \in (0, 2\pi)$. Show that $u(z) \le 0$. The function in part 1 is useful here.

3. Better version: Let u be a real-valued harmonic function in the disc $|z| < 1$ such that $u(z) \le M < \infty$ and $\lim_{r \to 1} u(re^{i\theta}) \le 0$ for almost all θ. Show that $u(z) \le 0$.

Proof, Part 1

Let $\phi : \mathbb{D} \to \mathbb{H}$ be the usual conformal mapping. Then $v = \operatorname{Im} \phi$ works.

Written out, $\phi(z) = i\frac{z+1}{z-1}$.

Proof, Part 2

Fix $0 < r < 1$ and apply the Poisson integral formula:

$$u(rse^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - r^2 s^2}{\left|re^{it} - rse^{i\theta}\right|^2} u(re^{it})\, dt$$

For a fixed s and θ, let the integrand be $g_r(t)$.

Our goal is to show that as $r \to 1$, $g_r(t)$ has integral going to 0. Note that $g_r(t) \to 1$ pointwise almost everywhere.

Furthermore, for $r > s$, we have that the Poisson kernel is bounded (it's continuous!) so $|g_r(t)| \lesssim 1$ for each fixed s as $1 \leftarrow r > s + \varepsilon$. Better yet, $\left|re^{it} - rse^{i\theta}\right|^2$ is bounded away from 0 because $s < 1$. This is good enough.

Dominated convergence gives us the desired result here.

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Problem 1

Let $f, f_n \in L^1$ with

1. $f_n \to f$ μ–ae.

2. $\|f_n\|_{L^1} \to \|f\|_{L^1}$.

Show then that $\|f_n - f\|_{L^1} \to 0$.

Proof

This is a standard result. It's a Fatou trick.

Note that $|f| + |f_n| - |f - f_n| \geq 0$. Thus we can apply Fatou to get

$$\begin{aligned} \int \liminf |f| + |f_n| - |f - f_n| &\leq \liminf \int |f| + |f_n| - |f - f_n| \\ 2\|f\|_{L^1} - 0 &\leq 2\|f\|_{L^1} - \liminf \|f - f_n\|_{L^1} \\ \liminf \|f - f_n\|_{L^1} &\leq 0 \end{aligned}$$

and thus $f_n \to f$.

Problem 2

Let μ be a finite positive Borel measure on $\mathbb{R}$ that is singular with respect to the Lebesgue measure. Show that

$$\lim_{r \to 0^+} \frac{\mu([x - r, x + r])}{2r} = +\infty$$

for μ–ae $x \in \mathbb{R}$.

Proof

This limit is usually equal to the Radon–Nikodym derivative when $\mu \ll m$, the Lebesgue measure.

Suppose there exists E with $\mu(E) \neq 0$ where $\lim_{r \to 0} \frac{\mu([x-r,x+r])}{2r} < \infty$. Let this limit be $f(x)$, where $f : E \to \mathbb{R}$ is finite on E.

Let A be such that $\mu(A^c) = 0$ and $m(A) = 0$. Define $E_n = \{x \in E \mid f(x) < n\} \cap A$.

Then intervals witnessing $m(E_n) = 0$ also witness $\mu(E_n) = 0$ if we take them small enough. I'm leaving this slightly informal for now, but I hope the details are clear.

Problem 3

If X is a compact metric space, $\mathscr{P}(X)$ is the set of Borel probability measures on X.

1. Let $\phi : X \to [0, \infty]$ be lower semicontinuous. Show that if $\mu, \mu_n \in \mathscr{P}(X)$ with $\mu_n \xrightharpoonup{*} \mu$, then
$$\int \phi \, d\mu \leq \liminf_{n \to \infty} \int \phi \, d\mu_n$$
2. Let $K \subseteq \mathbb{R}^d$ be a compact set. For $\mu \in \mathscr{P}(K)$, define
$$E(\mu) = \int_K \int_K \frac{1}{|x-y|} \, d\mu(x) \, d\mu(y)$$
Show that $E : \mathscr{P}(K) \to [0, \infty]$ attains its minimum on $\mathscr{P}(K)$ (which could possibly be ∞)

Proof, Part 1

This is part of the portmanteau theorem. Write $\phi = \lim f_n$ where the $f_n \geq 0$ are increasing, continuous, and converge pointwise.

Then by Montone Convergence,
$$\begin{aligned}
\int \phi \, d\mu &= \int \lim_n f_n \, d\mu \\
&\leq \lim_n \int f_n \, d\mu \\
&= \lim_n \lim_k \int f_n \, d\mu_k \\
&\leq \liminf_k \lim_n \int f_n \, d\mu_k \\
&= \liminf_k \int \phi \, d\mu_k
\end{aligned}$$
where we swap limits by the following argument.

Suppose $\{a_{n,k}\}$ are all non-negative, increasing in n, and $\lim_n \lim_k a_{n,k}$ exists. We want to show that $\lim_n \lim_k a_{n,k} \leq \liminf_k \lim_n a_{n,k}$.

Let N be such that $\left|\lim_k a_{n,k} - \lim_n \lim_k a_{nk}\right| < \varepsilon$ for all $n > N$, so in particular, $\lim_k a_{n,k} > \lim_n \lim_k -\varepsilon$.

Then for each k, $\lim_n a_{n,k} \geq a_{N,k}$. Then $\liminf_k \lim_n a_{n,k} \geq \liminf_k a_{N,k} > \lim_n \lim_k -\varepsilon$. Sending $\varepsilon \to 0$ gives the desired result.

Proof, Part 2

Let's show that E attains its minimum on $\mathscr{P}(K)$.

Let μ_n be a sequence of measures such that $E(\mu_n) \to \inf E = m$. Then there is a weakly-* convergent subsequence by the following argument.

Each μ_n is in the unit ball of $C(K)^*$, which is weakly-* compact. Furthermore, since $C(K)$ is separable, this is weakly-* sequentially compact too. Integrating against 1, we see that any weak-* limit is also a probability measure. Thus up to a subsequence $\mu_n \overset{*}{\rightharpoonup} \mu$.

Our first goal is to show that $\mu_n \otimes \mu_n \overset{*}{\rightharpoonup} \mu \otimes \mu$.

Let $f \in C(K \times K)$. If $f(x, y) = g(x)h(y)$, then we're done. Recall that the space of functions of the form $g(x)h(y)$ is dense in $C(K \times K)$ by Stone–Weierstrass.

Thus for $f_k(x, y) = g_k(x)h_k(y)$ with $f_k \to f$ uniformly, we have that

$$\int f_k d(\mu_n \otimes \mu_n) \to \int f_k d(\mu \otimes \mu)$$

and $\int f_k - f d(\mu_n \otimes \mu_n) < \|f_k - f\|_{L^\infty} \to 0$, so an application of the triangle inequality is all we need.

Finally, note that $1/|x - y|$ is lower semicontinuous, so

$$\int \frac{1}{|x-y|} d(\mu \otimes \mu) \leq \liminf_{n\to\infty} \int \frac{1}{|x-y|} d(\mu_n \otimes \mu_n) = m$$

and we're done.

Problem 4

Let $X = [0, 1]$ and $L^1 = L^1(X)$, $L^2 = L^2(X)$. then $L^2 \subseteq L^1$.

Show that L^2 is a meager subset of L^1, i.e., L^2 is a countable union of sets in L^1 that are closed and have empty interior in L^1.

Proof

It's enough to show that $\{f \in L^2 \mid \|f\|_{L^2} \leq 1\}$ is nowhere dense in L^1 and closed.

First take $f_n \to f$ in L^1 with $\|f_n\|_{L^2} \leq 1$. Is $f \in L^2$ with $\|f\|_{L^2} \leq 1$? Take a subsequence $f_{n_k} \to f$ pointwise almost everywhere. Then Fatou guarantees

$$\int |f|^2 \leq \liminf \int |f_{n_k}|^2 \leq 1$$

as desired.

Next, let's show that this set is nowhere dense. Consider $g(x) = x^{-1/2+\varepsilon}$. Then

$$\int_0^1 x^{-1/2+\varepsilon}\,dx = \frac{1}{1/2+\varepsilon}$$
$$\int_0^1 x^{-1+\varepsilon}\,dx = \frac{1}{\varepsilon}$$

Thus we have functions whose L^1 norms are bounded, but whose L^2 norms diverge. Adding a scaled version of a function like this to any member of $F = \{f \in L^2 \mid \|f\|_{L^2} \leq 1\}$ guarantees the existence of something outside F which is close in L^1 norm. Thus these sets are nowhere dense.

Problem 5

Let $X = C([0,1])$ be the Banach space of real-valued continuous functions on $[0,1]$ equipped with the sup-norm. Let $\mathscr{A}$ be the Borel σ-algebra on X.

Show that $\mathscr{A}$ is the smallest σ-algebra on X that contains all sets of the form

$$S(t,B) = \{f \in X \mid f(t) \in B\}$$

where $t \in [0,1]$ and $B \subseteq \mathbb{R}$ is a Borel set in $\mathbb{R}$.

Proof

See Spring 2017, Problem 3.

Let F be the σ-algebra on X generated by the sets $S(t,B)$. Just as easily, F could be generated by the sets $S(t,U)$ where U is open, since $S(t,U^c) = S(t,U)^c$ and $S(t,\bigcap U_\alpha) = \bigcap S(t,U_\alpha)$.

Let's show that F contains the open sets. Let $B(f,r)$ be open. Then for each $q \in [0,1] \cap \mathbb{Q}$, consider $S(q, B(f(q), r/2))$. Their intersection is a subset of $B(f,r)$, so by taking some unions, we get that $B(f,r)$ is contained in F. This ultimately gives us that $F \supseteq \mathscr{A}$

Next we need to show that $F \subseteq \mathscr{A}$. In other words, $\mathscr{A}$ must contain all the sets of the form $S(t,B)$.

But this is because $S(t,U)$ is open for U open.

Problem 6

Consider ℓ^1 and ℓ^∞. There is a well-defined dual pairing between them given by

$$\langle u, v\rangle = \sum u_i v_i$$

for $u \in \ell^1$ and $v \in \ell^\infty$. With this dual pairing, $\ell^\infty = (\ell^1)^*$.

1. Show that there exists no sequence $\{u_n\} \subseteq \ell^1$ such that

   (a) $\|u_n\|_{\ell^1} \geq 1$ for all $n \in \mathbb{N}$ and

   (b) $\langle u_n, v\rangle \to 0$ for each $v \in \ell^\infty$

2. Show that every weakly convergent sequence $\{u_n\}$ in ℓ^1 converges in the norm topology.

Proof, Part 1

Suppose $x^n \rightharpoonup 0$. Let's show $\|x^n\| \to 0$. Suppose $\|x^n\| \geq 1$. We'll show there's a subsequence $\{x^{n_k}\}$ with $\langle y, x^{n_k}\rangle > 1/3$ for some $y \in \ell^\infty$.

Let n_0 be such that $\sum_{n \leq n_0} \left|x_n^0\right| \geq 1/3$. Then For $0 \leq k \leq n_0$, take $y_k = \operatorname{sign} x_k^0$.

Then $\left\langle y, x^0\right\rangle > 1/3$.

Next, pick k_1 such that $\sum_{n \leq n_0} \left|x_n^{k_1}\right| < 1/6$ and $\sum_{n_0 \leq n \leq n_1} \left|x_n^{k_1}\right| \geq 1/3$.

Then for $n_0 < j \leq n_1$, take $y_j = \operatorname{sign} x_j^{k_1}$. Note that $\left\langle y, x^0\right\rangle > 1/3$ and $\left\langle y, x^{k_1}\right\rangle > 1/3$.

Inducting, we get $\left\langle y, x^{k_j}\right\rangle > 1/3$ for all j, a contradiction to weak convergence to zero.

Proof, Part 2

Suppose $x_n \rightharpoonup x$. Then $\langle x_n - x, v\rangle \to 0$ for all v, so $\|x_n - x\|$ cannot be bounded below by 1. In particular (by rescaling), it cannot be bounded below by any positive constant. By considering subsequences, it must in fact go to zero. Thus $x_n \to x$ strongly.

Problem 7

Let $\mathscr{H}$ be the space of holomorphic functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f|^2 \, dA(z) < \infty.$$

The vector space $\mathscr{H}$ is equipped with the L^2 inner product. Fix $z_0 \in \mathbb{D}$ and define $L_{z_0}(f) = f(z_0)$ for $f \in \mathscr{H}$.

1. Show that $L_{z_0} : \mathscr{H} \to \mathbb{C}$ is a bounded linear functional on $\mathscr{H}$.

2. Find an explicit function $g_{z_0} \in \mathcal{H}$ such that

$$L_{z_0}(f) = f(z_0) = \left\langle f, g_{z_0} \right\rangle$$

for all $f \in \mathcal{H}$.

The final result is *Bergman's kernel formula*. See p161 in Ahlfors.

Proof, Part 1

This is clearly a linear map. Let $B(z_0, r) \subseteq \mathbb{D}$. By the mean value property,

$$\begin{aligned} f(z_0) &= \frac{1}{\pi r^2} \int_{B(z_0,r)} f(z_0) \\ |f(z_0)| &\leq \frac{1}{\pi r^2} \sqrt{\pi r^2} \|f\|_{\mathcal{H}} \end{aligned}$$

so it's bounded too.

Proof, Part 2

See Spring 2018, #12.

First, note that $\{e_n := \sqrt{\frac{n+1}{\pi}} z^n\}$ is an orthonormal basis for $\mathcal{H}$. Orthonormality is easy. To show it's a basis, suppose $\langle f, e_n \rangle = 0$ for all n.

We compute

$$\begin{aligned} \langle f, e_n \rangle &= \sqrt{\frac{n+1}{\pi}} \int_{\mathbb{D}} f(re^{i\theta}) r^{n+1} e^{-in\theta} \, dr \, d\theta. \\ &= \sqrt{\frac{n+1}{\pi}} \int_{\mathbb{D}} r^{2n+1} \frac{f(re^{i\theta})}{r^{n+1} e^{i(n+1)\theta}} re^{i\theta} \, d\theta \, dr \\ &= \sqrt{\frac{n+1}{\pi}} \int_0^1 r^{2n+1} f^{(n)}(0) \, dr \\ &= c_n f^{(n)}(0) \end{aligned}$$

by Cauchy's diffentiation formula.

Thus if $\langle f, e_n \rangle = 0$ for all n, then $f = 0$.

Now, to determine g_w for some $z \in \mathbb{D}$ we have

$$\begin{aligned} g_w(z) = \langle g_w, g_z \rangle &= \sum \langle g_w, e_n \rangle \overline{\langle g_z, e_n \rangle} \\ &= \sum \overline{e_n(w)} e_n(z) \\ &= \sum_{n=0}^{\infty} \frac{n+1}{\pi} (\overline{w} z)^n \\ &= \frac{1}{\pi(1-\overline{w}z)^2} \end{aligned}$$

by differentiating the geometric series.

Problem 8

Let f be a holomorphic on $\mathbb{D}$ and continuous up to the boundary. with $f(0) \neq 0$.

1. Show that if $0 < r, 1$ and $\inf_{|z|=r} |f(z)| > 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log\left| f(re^{i\theta}) \right| d\theta \geq \log|f(0)|$$

2. Show that $m\{\theta \in [0, 2\pi] \mid f(e^{i\theta}) = 0\} = 0$ where m is the Lebesgue measure.

Proof, Part 1

This is part of the Poisson-Jensen Formula. Let f have the zeroes $a_1, \ldots, a_n$ in $B(0,r)$. Define

$$B(z) = \prod_{i=1}^{n} \frac{r(z-a_i)}{r^2 - \overline{a_i} z}$$

which satisfies $|B(z)| = 1$ on $\partial B(0,r)$ and B and f share zeroes on $B(0,r)$.

Then $\log|f/B|$ is harmonic and so the mean value property gives us

$$\begin{aligned} \frac{|f(0)|}{|B(0)|} &= \fint_0^{2\pi} \log\left| f(re^{i\theta}) \right| - \log\left| B(re^{i\theta}) \right| d\theta \\ &= \fint_0^{2\pi} \log\left| f(re^{i\theta}) \right| d\theta \end{aligned}$$

Proof, Part 2

Define $E = \{\theta \mid f(e^{i\theta}) = 0\}$. Suppose $mE = \alpha > 0$. Then for each $\theta \in E$, we have by continuity that $f(re^{i\theta}) \to 0$ as $r \to 1$.

Since $f(re^{i\theta}) \leq M$ on $\mathbb{D}$, we have that $\log\left|f(re^{i\theta})\right| \leq \log|M| < \infty$ on the disc.

Let $g_r(\theta) = M - \log\left|f(re^{i\theta})\right|$. Then we can apply Fatou's lemma to get:

$$\fint_0^{2\pi} M - \log\left|f(e^{i\theta})\right| \leq M - \limsup \fint_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta$$
$$f(0) \leq \limsup \fint_0^{2\pi} \log\left|f(re^{i\theta})\right| \leq \fint \log\left|f(e^{i\theta})\right| = -\infty,$$

a contradiction.

Problem 9

Let μ be a Borel probability measure on $[0, 1]$.

1. Show that f defined as
$$f(z) = \int_{[0,1]} e^{izt}\, d\mu(t)$$
is holomorphic on $\mathbb{C}$.

2. Suppose there exists $n \in \mathbb{N}$ such that
$$\limsup_{|z|\to\infty} \frac{|f(z)|}{|z|^n} < \infty.$$
Show then that $\mu = \delta_0$, the Dirac delta.

Proof, Part 1

Both parts follow from considering μ a tempered distribution and applying a Paley-Weiner theorem.

In any case, analyticity follows by differentiation under the integral sign. Let's do it.

$$\begin{aligned} \frac{f(z+w) - f(z)}{w} &= \frac{\int_0^1 e^{i(z+w)t}\, d\mu(t) - \int_0^1 e^{izt}\, d\mu(t)}{w} \\ &= \int_0^1 e^{izt} \frac{e^{iwt} - 1}{w}\, d\mu(t) \end{aligned}$$

Notice that $(e^{iwt} - 1)/w \to it$ as $w \to 0$. Thus the integrand is bounded by $2|e^{izt}|$ for small enough w. Thus dominated convergence guarantees

$$f'(z) = \int_0^1 ite^{izt}\, d\mu(t)$$

Proof, Part 2

Note that f is holomorphic. Since f is bounded by some polynomial, it must itself then by a polynomial.

Next, we notice that for real z, we have $|f(z)| \leq \int_0^1 |e^{izt}|\, d\mu(t) \leq 1$. Thus f is bounded by 1 on the positive real axis. The only polynomial that does this is a constant. Thus f is constant.

Note that $f(0) = 1$ also, so $f(z) = 1$ everywhere. By Fourier inversion on tempered distributions, μ must be equal to a measure with the same Fourier transform. Thus $\mu = \delta_0$.

Alternatively, note that $\int_0^1 e^{izt}\, d\mu(t)$ is real for all t given some real z. This means that $\mu(t \neq \pi k/z) = 0$. This holds for all k and real z, so we can force $\mu(0) = 1$.

Problem 10

Consider the quadratic polynomial $f(z) = z^2 - 1$ on $\mathbb{C}$. We are interested in the iterates f^n where $f^0 = \text{id}$ and $f^{n+1} = f \circ f^n$.

1. Find an explicit constant $M > 0$ such that the following dichotomy holds for each $z \in \mathbb{C}$: either

   (a) $|f^n(z)| \to \infty$ as $n \to \infty$

   (b) $|f^n(z)| \leq M$ for all $n \in \mathbb{N}_0$.

2. Let U be the set of all $z \in \mathbb{C}$ for which the first alternative 1. holds and K the set of all $z \in \mathbb{C}$ for which the second holds.

   Show that U is open and K is compact without "holes", i.e., $C \setminus K$ has no bounded connected components.

Proof, Part 1

We don't have to find the minimal constant, so 2 trivially works. (Pick 100 if you want even).

Proof, Part 2

Consider $U_k = \{z \mid \left|f^k(z)\right| > 2\}$. This is open. Then U is the union of open sets which is thus open. In particular, K is closed (and bounded).

Suppose S was a bounded connected component of U. Then $f^k(x) < M$ for all $x \in K$, so in particular for all $x \in \partial S$. But then $f^k(x)$ is bounded by M inside S by the maximum principle.

Problem 11

Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic with $z \mapsto g(z) = f(z)f(1/z)$ bounded on $\mathbb{C} \setminus \{0\}$.

1. Show that if $f(0) \neq 0$, then f is constant.
2. Show that if $f(0) = 0$, then there exists $n \in \mathbb{N}$ and $a \in \mathbb{C}$ such that $f(z) = az^n$ for all $z \in \mathbb{C}$.

Proof, Part 1

Suppose g is bounded by M. Suppose $f(z) \neq 0$. Then $|f(z)| > m$ on some δ-neighborhood of 0. Thus for $|z| < \delta$, we have $M \geq f(z)f(1/z) \geq mf(1/z)$.

Thus $f(1/z) \leq M/m$ in a neighborhood of ∞. Thus f is constant.

Proof, Part 2

Suppose $f(0) = 0$. Write $f(z) = z^n h(z)$. Then $h(z)$ is constant by the above and we're done.

Problem 12

Let $U \subseteq \mathbb{C}$ be an open set and $K \subseteq U$ a compact subset.

1. Prove that there exists bounded open V with $K \subseteq V \subseteq \overline{V} \subseteq U$ such that ∂V consists of finitely many closed line segments.
2. Let f be a holomorphic function on U. Show that there exists a sequence $\{R_n\}$ of rational functions such that

   (a) $R_n \to f$ uniformly on K, and

   (b) none of the functions R_n has a pole in K.

   Hint: First represent $f(z)$ for $z \in K$ as a suitable integral over the set ∂V and then notice that the integrand is equicontinuous in z.

Proof, Part 1

This is trivial.

Proof, Part 2

Let V be as above. For any $z \in K$, we write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial V} \frac{f(w)}{w-z}\, dw \\ &= \frac{1}{2\pi i} \sum_{j=1}^{N} \int_{\gamma_j} \frac{f(w)}{w-z}\, dw \\ &= \frac{1}{2\pi i} \sum_{j=1}^{N} \int_0^1 \frac{f(\gamma_j(t))\gamma_j'(t)}{\gamma_j(t)-z}\, dt \end{aligned}$$

where the γ_j's are all straight lines with the same length.

Let's try to write f as a rational function. To do this, we approximate the integral by a Riemann sum. Note that $\left|\gamma_j'(t)\right| = c$ for some constant c and all j.

Pick a fine mesh and then try to evaluate

$$\left| f(z) - \sum_{j=1}^{N} \sum_{i=1}^{M} \frac{f(\gamma_j(t_i))\gamma_j'(t_i)}{\gamma_j(t_i) - z}(t_i - t_{i-1}) \right|$$

Everything works as expected and note that the poles are only on γ_j, i.e., outside of K.

11 S17

Problem 1

Let $K \subseteq \mathbb{R}$ be a compact set of positive measure,and let $f \in L^\infty(\mathbb{R})$. Show that

$$F(x) = \frac{1}{|K|} \int_K f(x+t)\, dt$$

is *uniformly* continuous on $\mathbb{R}$.

Solution

We calculate

$$\begin{aligned}|F(x) - F(y)| &= \frac{1}{|K|}\left|\int_K f(x+t) - f(y+t)\,dt\right| \\ &= \frac{1}{|K|}\left|\int_{K-x} f(t)\,dt - \int_{K-y} f(y)\,dt\right|\end{aligned}$$

For continuity, I'd just say $\left|\int_K f(x+t) - f(y+t)\,dt\right| \le |K|\left\|f - \tau_{y-x} f\right\|_{L^\infty}$, which locally goes to zero. But translations are not continuous in L^∞, so we need better.

We continue:

$$\begin{aligned}|F(x) - F(y)| &\le \frac{1}{|K|}\int_{(K-x)\Delta(K-y)} |f(t)|\,dt \\ &\le \frac{\|f\|_{L^\infty}}{|K|}|(K-x)\Delta(K-y)|\end{aligned}$$

Write $h = x - y$. Then $|(K-x)\Delta(K-y)| = |(K-h)\Delta K|$. Written like this, we're ready for uniform convergence!

Let $\varepsilon > 0$. Pick $V \supseteq K$ open with $|V \setminus K| < \varepsilon/4$. Write $V = I_1 \cup \cdots \cup I_n$ as a disjoint union of open intervals. Then

$$(K-h)\Delta K = ((K-h) \setminus K) \cup (K \setminus (K-h))$$

Then $(K-h) \setminus K \subseteq (V-h) \setminus K \subseteq ((V-h) \setminus V) \cup (V \setminus K)$

Thus

$$\begin{aligned}|(K-h)\Delta K| &\le 2|V \setminus K| + |((V-h) \setminus V) \cup (V \setminus (V-h))| \\ &< \varepsilon/2 + |(V-h)\Delta V|\end{aligned}$$

This was basically just to get V instead of K.

For small enough h relative to the smallest interval of V, $|(V-h)\Delta V| = 2n|h|$. Thus for $|h| < \varepsilon/4n$, we get $|F(x) - F(y)| \le \frac{\|f\|_{L^\infty}}{|K|}\varepsilon$.

Since n depends only on ε and K, this shows uniform continuity.

Alternate (cleaner) proof

We calculate (letting $y - x = h$)

$$\begin{aligned}
F(x) - F(y) &= \frac{1}{|K|}\int_K f(x+t) - f(y+t)\,dt \\
&= \frac{1}{|K|}\left(\int_K f(x+t)\,dt - \int_K f(x+t+(y-x))\,dt\right) \\
&= \frac{1}{|K|}\left(\int_K f(x+t)\,dt - \int_{K-(y-x)} f(x+t)\,dt\right) \\
&= \frac{1}{|K|}\int_{K\cap(K-h)} f(x+t) - f(x+t)\,dt + \frac{1}{|K|}\int_{K\triangle(K-h)} \pm f(x+t)\,dt \\
|\mathrm{LHS}| &\le \frac{1}{|K|}\int_{K\triangle(K-h)} |f(x+t)|\,dt \\
&\le \|f\|_{L^\infty}|K \triangle (K-h)|
\end{aligned}$$

where $\triangle$ is the symmetric difference and the $\pm$ sign depends on whether we're in $K \setminus (K-h)$ or $(K-h) \setminus K$.

But note that $|K \triangle (K-h)| = \|\chi_K - \tau_h\chi_K\|_{L^1}$, which goes to zero as $h \to 0$ in L^1 by continuity of the translation operator.

Note: If we're careless about estimates, we will get $\int_K f - \tau_{y-x}f$ and try to bound this by $|K|\left\|f - \tau_{y-x}f\right\|_{L^\infty}$, which is dumb, since the translation operators are certainly not continuous in L^∞.

Better Understanding

Write $g(x) = f(-x)$ and then

$$\begin{aligned}
F(x) &= \frac{1}{|K|}\int_K f(x+t)\,dt \\
&= \left(g * \frac{1}{K}\chi_K\right)(-x)
\end{aligned}$$

which is the convolution of an L^1 and L^∞ function, and hence uniformly continuous. Continuity is trivial by Young's Inequality. Uniform continuity is proven by approximating the L^1 component by something in C_c.

Problem 2

Let $f_n : [0,1] \to [0,\infty)$ be a sequence of functions, each non-decreasing on $[0,1]$. Suppose f_n is uniformly bounded in L^2. Show that there exists a subsequence converging in L^1.

Proof

Banach-Alaoglu is a red herring

If f_n were a characteristic function on $(t,1)$, then it would have height at most $M/\sqrt{1-t}$, so that it would have L^2 norm $\sqrt{(1-t)M^2/(1-t)} = M$. Since the f_n are all non-decreasing, if they reach the value $\sqrt{M}/\sqrt{1-t}$ at t, they must be at least as large as the characteristic function described above, so in fact

$$0 \le f_n(t) \le \frac{M}{\sqrt{1-t}}$$

For fixed t, $f_n(t)$ is in a compact set, and so we find f_{n_k} convergent on $[0,1] \cap \mathbb{Q}$ by diagonalization.

Then in fact f_{n_k} converge pointwise a.e.. Let $q \in \mathbb{Q}$ and let $a_q = \lim f_{n_k}(q)$. We have $a_q \le a_{q'}$ for $q < q'$. For $r \in \mathbb{R}$, let $L_r = \sup_{q<r} a_q$ and $U_r = \inf_{q'>r} a_{q'}$.

Both $r \mapsto L_r$ and $r \mapsto U_r$ are continuous almost everywhere being non-decreasing. Thus their shared continuity points are almost everywhere, so $f_{n_k}(r)$ converges ae. (in fact everywhere but a countable set).

Now $f_{n_k} \to f$ pointwise a.e. We want L^1 convergence. We have $\left|f_{n_k}(t) - f(t)\right| \le M/\sqrt{1-t}$ for a.e. t. This is in L^1 so dominated convergence gives us $f_{n_k} \to f$ in L^1.

Rewritten Proof

Note that each f_n is continuous except on a countable set. Thus outside of some fixed N, every f_n is continuous.

Fix $0 \le t < 1$ and suppose $f(t) = \alpha$. Then $M^2 \ge \int_\alpha^1 |f(x)|^2\,dx \ge |1-t|\alpha^2$. Thus

$$f(<t) \le f(t) = \alpha \le \frac{M}{\sqrt{1-t}}$$

with a horrible abuse of notation which is nevertheless clear.

Now outside of N, at each point of a dense set $D \subseteq [0,1] \setminus N$, make sure we find some subsequence converging pointwise. We end up (after diagonalization), getting that some f_{n_k} converges pointwise outside of N on a dense set D.

The claim now is that f_{n_k} converges pointwise everywhere outside of N. Note that $\lim f_{n_k}\restriction_D$ is monotone on D. Thus there is a montone extension to N^c. At a point of continuity of this montone extension, showing f_{n_k} converges pointwise here is easy.

Finally, we have pointwise convergence and a bound by $M/\sqrt{1-t}$. This function is L^1, so dominated convergence finishes the job.

Remarks

This is in general false for $f_n : [0,1] \to \mathbb{R}$ if we disregard the non-decreasing condition. (Compact embeddings are hard to come by.)

Consider the functions f_n which alternate between ± 1 at a frequency 2^{-n}. These are bounded in L^∞, but clearly have no convergent subsequence in L^1.

Problem 3

Let $C([0,1])$ be endowed with the sup norm. Let $\mathcal{F}$ be a σ-algebra on $C([0,1])$ such that $L_x(f) = f(x)$ is $\mathcal{F}$-measurable. Show that $\mathcal{F}$ contains all open sets.

Solution

Let's show that $\mathcal{F}$ contains closed sets (or rather, contains closed balls). Let $B = \{f \mid \|f - g\| \le \varepsilon\}$. Then for each $q \in \mathbb{Q} \cap [0,1]$, let

$$E_q = \{f \in C \mid |f(q) - g(q)| \le \varepsilon\} = L_q^{-1}(\overline{B(g(q), \varepsilon)})$$

Then $B = \bigcap_q E_q$ by continuity, so $B \in \mathcal{F}$.

Problem 4

For $n \ge 1$, let $a_n : [0,1) \to \{0,1\}$ denote the n-th digit in the binary expansion of x, so that

$$x = \sum_{n \ge 1} a_n(x) 2^{-n}$$

for all $x \in [0,1)$. (Remove ambiguity by requiring that $\liminf a_n = 0$.)

Let $M = M([0,1))$ denote the Banach space of finite complex Borel measures on $[0,1)$ and define linear functions L_n on M via

$$L_n(\mu) = \int_0^1 a_n(x)\, d\mu(x)$$

Show that no subsequence of $\{L_n\}$ converges in the weak-$\star$ topology on M^*.

Solution

Consider $\mu = \delta_{0.10101\ldots}$. Then $L_n(\mu)$ does not converge.

Thus given any subsequence L_{n_k}, we can just take δ_x where x has 1 in the n_{2k} spot and 0 in the n_{2k+1} spot.

Problem 5

Let μ be a finite complex Borel measure on $[0,1]$ such that $\hat{\mu}(n) \to 0$ as $n \to \infty$. Let $\nu \ll \mu$ be another finite complex Borel emasure. So that $\hat{\nu}(n) \to 0$ as $n \to \infty$ also.

Solution

Radon-Nikodym guarantees $\nu(E) = \int f\, d\mu$ for some $f \in L^1(\mu)$. Fix $\varepsilon > 0$. Select g continuous with $\|f - g\|_{L^1} < \varepsilon$. Next, select P a trigonometric polynomial with $\|g - P\|_{L^\infty} < \varepsilon$.

Note that $\int e^{2\pi i n x} P\, d\mu \to 0$ since eventually n is too high a frequency.

Problem 6

Let $\overline{\mathbb{D}}$ be the closed unit disc in the complex plane, let $\{p_n\}$ be distinct points in $\mathbb{D}$ and let $r_n > 0$ be such that the discs $D_n = B(p_n, r_n)$ are disjoint, contained in $\mathbb{D}$, and $\sum r_n < \infty$.

Prove that $X = \overline{D} \setminus (\bigcup_{\mathbb{N}} D_n)$ has positive area.

Solution

Let $f(x) = \sum \chi_{\pi_1(D_n)}(x)$. Note that $f(x)$ counts the number of discs above x.

So then $\int_{-1}^{1} f(x) = \sum 2r_n < \infty$ by Fubini or something. So f is finite ae, so the number of discs above every point is finite ae. Thus there's some 1D mass above each point, so there's some area in the complement.

Problem 7

Let $f : \{1 \leq |z| \leq R\} \to \{1 \leq |z| \leq S\}$ be injective, continuous on the closed annulus, and injective on the interior. Show that $S = R$.

Solution

Without loss of generality, f sends the unit circle to itself. Since f is non-vanishing analytic, $\log|f|$ is harmonic and satisfies $\log|f(z)| = 0$ on the circle, and $\log|f(z)| = \log(S)$ on $|z| = R$.

Since we can solve the Dirichlet problem on the annulus, $\log|z|$ is uniquely determined by its boundary values, and since $\frac{\log(S)}{\log(R)} \log|z|$ is another harmonic function with the same boundary values, they're equal.

$$\log|f(z)| = \frac{\log(S)}{\log(R)}\log|z|$$

and so $|f(z)| = |z|^{\log(S)/\log(R)} = \left|z^{\log S/\log R}\right|$.

Since f and z^α are both analytic (taking a branch cut), and have the same absolute value, they're equal up to rotation. Thus $f = Cz^\alpha$ where $\alpha = \log S/\log R$. But they're injective, so $\alpha = 1$, so $S = R$ and in fact f is a rotation.

Problem 8

Let $a_1, \dots, a_n$ be $n \geq 1$ points in the disc $\mathbb{D}$ (possibly with repetitions), so that

$$B(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j}z}$$

has n zeroes in $\mathbb{D}$. Prove that the derivative $B'(z)$ has $n-1$ zeroes in $\mathbb{D}$.

Proof

First, let's do this for $B(0) \neq 0 \neq B'(0)$ and B without repeated roots. Then

$$\begin{aligned}\frac{B'}{B} &= \sum \frac{1 - \left|a_j\right|^2}{(z - a_j)(1 - \overline{a_j}z)} \\ &= \frac{\sum_j \left((1 - |a_j|^2)\prod_{i\neq j}(z - a_i)(1 - \overline{a_i}z)\right)}{\prod_j (z - a_j)(1 - \overline{a_j}z)}\end{aligned}$$

Check (with actually little effort) that $\overline{B'/B}(1/\bar{z}) = z^2(B'/B)(z)$, and so $B'(z) = 0$ iff $B'(1/\bar{z}) = 0$, and so the roots come in pairs, one inside the disc and one outside.

Since B'/B shares zeroes with B', and B'/B has a numerator of degree $2(n-1)$, it has $2(n-1)$ total zeroes, and thus $n-1$ in the disc.

For the general case, take $B_k \to B$ that look basically the same but don't have repeated roots, aren't zero at 0, don't have deriative zero at 0, and converge uniformly to B. The result then holds by saying that

$$\text{number of zeroes of } B' = \int_{\partial\mathbb{D}} \frac{B''}{B'} = \lim \int_{\partial\mathbb{D}} \frac{B_k''}{B_k'} = n - 1$$

by the argument principle.

Problem 9a

Let f be an analytic function in the entier complex plane $\mathbb{C}$ and assume $f(0) \neq 0$. Let $\{a_n\}$ be the zeroes of f, repeated according to their multiplicities. Let $R > 0$ be such that $|f(z)| > 0$ on $|z| = R$. Prove

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\theta})\right| d\theta = \log|f(0)| + \sum_{|a_n|<R} \log\frac{R}{|a_n|}$$

Proof

(This is Jensen's formula).

There are only finitely many zeroes in $|z| < R$ since f isn't identically zero. Thus we can define

$$g(z) = \prod_{|a_n|<R} \frac{R(z-a_n)}{R^2 - \overline{a_n}z}$$

This is a Blaschke (scaled accordingly). It shares the same zeroes as f, has no poles, and satisfies $|g(z)| = 1$ on $|z| = R$. Thus f/g is nonvanishing holomorphic on the disc $B(0,R)$ and $|f/g| = |f|$ on the boundary. Thus $\log|f/g|$ is harmonic. Apply the mean value formula.

$$\log\left|\frac{f(0)}{g(0)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(Re^{i\theta})}{g(Re^{i\theta})}\right| d\theta = \fint \log\left|f(Re^{i\theta})\right| d\theta$$

Calculate

$$\log\left|\frac{f(0)}{g(0)}\right| = \log|f(0)| - \sum_{|a_n|<R} \log\left|\frac{R(0-a_n)}{R^2-0}\right| \quad = \log|f(0)| + \sum \log\left|\frac{R}{a_n}\right|$$

and we're done.

Problem 9b

Prove that if there are constants C and λ such that $|f(z)| \leq Ce^{|z|^\lambda}$ for all z, then

$$\sum \left(\frac{1}{|a_n|}\right)^{\lambda+\varepsilon} < \infty$$

for all $\varepsilon > 0$.

Proof

See Ahlfors, page 210.

Let $N(R) = \#\{n \mid |a_n| < R\}$. The above part tells us that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(2Re^{i\theta})\right| d\theta &= \log|f(0)| + \sum_{|a_n|<2R} \log\left(\frac{2R}{|a_n|}\right) \\ &\geq \log|f(0)| + \sum_{|a_n|<R} \log\left(\frac{2R}{|a_n|}\right) \\ &\geq \log|f(0)| + N(R)\log(2)\end{aligned}$$

since each term in the sum is at least log(2).

Furthermore,

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(2Re^{i\theta})\right| d\theta \leq (2R)^\lambda + \log(C)$$

These estimates give us

$$N(R) \leq \frac{(2R)^\lambda - \log(C) - \log|f(0)|}{\log 2} \leq K(2R)^\lambda$$

for some K and R big enough. Pick M such that this holds for $R \geq 2^{M-1}$. We just need to show that $\sum_{|a_n|\geq 2^{M-1}} \frac{1}{|a_n|^{\lambda+\varepsilon}}$ converges, since it's the tail.

On the tail, perform a telescoping series and bound $|a_n|$ by the R in $N(R)$. We write

$$\begin{aligned}\sum_{|a_n|\geq 2^{M-1}} \left(\frac{1}{|a_n|}\right)^{\lambda+\varepsilon} &= \sum_{r=M}^{\infty} \sum_{2^{r-1}\leq |a_n| < 2^r} \left(\frac{1}{|a_n|}\right)^{\lambda+\varepsilon} \\ &\leq \sum_{r=M}^{\infty} \left(N(2^r) - N(2^{r-1})\right)\left(\frac{1}{2^{r-1}}\right)^{\lambda+\varepsilon}\end{aligned}$$

Discard $N(2^{r-1})$ and we get a geometric series.

Rewritten Proof

Note that if $|f(z)| \leq Ce^{|z|^\lambda}$, then $\log|f| \lesssim |z|^\lambda$. Thus applying the first equation,

$$R^\lambda \gtrsim \sum_{|a_n|<R} \log\frac{R}{|a_n|}$$

If we limit our sum, we can guarantee each summand is large.

$$\begin{aligned} R^\lambda &\geq \sum_{|a_n|<R/2} \log \frac{R}{|a_n|} \\ &\geq \sum_{|a_n|<R/2} \log 2 \end{aligned}$$

Thus $\#\{a_n \in B(0, R/2)\} \lesssim R^\lambda$. Thus

$$\sum \left(\frac{1}{|a_n|}\right)^{\lambda+\varepsilon} \lesssim \sum_{R\sim 2^N} \left(\frac{1}{R}\right)^{\lambda+\varepsilon} R^\lambda < \infty$$

where we multiply the approximate value by the number of terms.

Problem 10

Let $a_1, \dots, a_n$ be $n \geq 1$ distinct points in $\mathbb{C}$ and let $\Omega = \mathbb{C} \setminus \{a_1, \dots, a_n\}$. Let $H(\Omega)$ be the vector space of real harmonic functions on Ω and $R(\Omega)$ be the space of real parts of analytic functions. Show that H/R has dimension n, find a basis, and prove it's a basis.

Proof

Try $f_i = \log|z - a_i|$ (think of integrating $1/z$ around the holes). Let $\gamma_1, \dots, \gamma_n$ be a homology basis, i.e., little CCW circles around each hole.

Let $u \in H$. Write $*du = -\partial_y u dx + \partial_x u dy$. The periods are $\int_{\gamma_i} *du$.

Note that $\log|z|$ has period 2π around 0.

Claim: if u is harmonic with conjugate differential having period zero, then write $f dz = du + i * du$. This has an antiderivative, since du is certainly exact. The antiderivative has real part u.

Since each f_i has period $2\pi\delta_{ij}$ around the curve γ_j, we can subtract off linear combinations of the f_i to get periodless functions.

Problem 11

Let $1 \leq p < \infty$ and let u be harmonic on $\mathbb{C}$ and in L^p. Prove that $u = 0$.

Proof

Classic application of mean value formula. Write

$$
\begin{aligned}
u(z) &= \frac{1}{\pi r^2} \int_{B(z,r)} u(\xi)\, d\xi \\
|u(z)| &\leq \frac{1}{\pi r^2} \int_{B(z,r)} |u(\xi)|\, d\xi \\
&\leq \frac{1}{\pi r^2} \|u\|_{L^p} \|1\|_{L^{p'}} \\
&\leq C \frac{\pi^{1/p'} r^{2/p'}}{\pi r^2} \to 0
\end{aligned}
$$

Problem 12

Let $0 < \alpha < 1$ and let $f(z)$ be analytic on $\mathbb{D}$. Prove that if f is α-Hölder continuous with constant C, then there exists $A = A(C) < \infty$ such that

$$
|f'(z)| \leq A(1 - |z|)^{\alpha - 1}
$$

Proof

Let's apply the Cauchy integral formula.

$$
\begin{aligned}
f'(z) &= \int_{\partial B(z,r)} \frac{f(w)}{(z-w)^2}\, dw \\
&= \int_{\partial B(z,r)} \frac{f(w) - f(z)}{(z-w)^2}\, dw
\end{aligned}
$$

by subtracting zero. Then we bound!

$$
\begin{aligned}
|f'(z)| &\leq \int_{\partial B(z,r)} \frac{|f(w) - f(z)|}{|z-w|^2}\, dw \\
&\leq C \int |z - w|^{\alpha - 2}\, dw \\
&\leq 2\pi C r^{\alpha - 2 + 1} = 2\pi C r^{\alpha - 1}
\end{aligned}
$$

Just need to pick r so that $B(z, r) \subseteq \mathbb{D}$, so take $r = (1 - |z|)/2$ and we're done.

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Problem 1

Suppose $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing. Show that if $A \subseteq \mathbb{R}$ is Borel, then so is $f(A)$.

Proof

Let $\mathcal{F} = \{A \subseteq \mathbb{R} \mid f(A) \text{ is Borel}\}$. We need to show $\mathcal{F}$ is a σ-algebra containing all closed intervals (or open).

We have $\varnothing \in \mathcal{F}$. Also $\mathbb{R}$ is in $\mathcal{F}$, since the image of f is a countable union of intervals (since atmost countably many jump discontinuities). Similarly, $f([a,b])$ is Borel.

Next, if $f(A)$ is Borel, note that $f(A)$ and $f(A^c)$ have at most countably many points in common, since f can be constant on at most countably many intervals (since each contains a rational number).

Unions are good too, so we're done.

Another perspective: This question is really asking if we can construct a pullback of $\mathcal{B}$ along f. Normally we can only construct pushforwards of σ-algebras.

Problem 2

Let $\{f_n\}$ be a bounded sequence in $L^2([0,1])$ which converges a.e.. Show that f_n converges in the weak topology on L^2.

Proof

Let $f_n \to f$ almost everywhere. First, Fatou guarantees that $f \in L^2$ (with the same norm bound).

Let's show $\int f_n g \to \int fg$ for all $g \in L^2$. Fix $g \in L^2$ and $\varepsilon > 0$. Let's use Egorov to get uniform convergence on a big set.

Pick $\delta > 0$ such that $|E| < \delta \implies \int_E |g|^2 < \varepsilon$, possible since $|g|^2$ is integrable. (Proof: consider $h_n = \min\{n, h\}$ for h integrable and positive).

Then by Egorov, there's some set E with $|E| < \delta$ and $f_n \to f$ on E^c.

$$\begin{aligned}\int |f_n g - fg| &\leq \|g\|_{L^2(E)} \|f_n - f\|_{L^2(E)} + \|g\|_{L^2(E^c)} \|f_n - f\|_{L^2(E^c)} \\ &\to \varepsilon M + 0\end{aligned}$$

so sending $\varepsilon \to 0$ gives us the result.

Problem 3

Let μ_n be a sequence of Borel probability measures on $\mathbb{R}$ and define $F_n(x) = \mu_n((-\infty, x])$.

Suppose F_n converge uniformly on $\mathbb{R}$. Show then for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, the numbers

$$\int f \, d\mu_n$$

converge as $n \to \infty$.

Proof

This is a weak convergence result. Let's show this first for \$$\chi_{\text{(a,b]}}$\$-like functions. Let $g = \sum_k \alpha_k \chi_{(a-k, b_k]}$.

$$\begin{aligned}\left|\int g \, d\mu_n - \int g \, d\mu_m\right| &= \left|\sum \alpha_k (F_n(b_k) - f_n(a_k)) - \sum \alpha_k (F_m(b_k) - F_m(a_k))\right| \\ &\leq \sum |\alpha_k| \left(|F_n(b_k) - F_m(b_k)| + |F_n(a_k) - F_m(a_k)|\right)\end{aligned}$$

which goes to zero by uniform convergence. (This is basically trivial).

Any bounded continuous function can be approximated in L^∞ by these functions on a compact interval (can't do it for $\mathbb{R}$ because you might need infinitely many functions).

Take a large enough interval that μ_n outside is small (possible by uniform convergence and that each μ_n is a probability measure). Then apply approximation.

Problem 4

Consider $V = C([-1, 1])$ a Banach space of real-valued continuous functions with the sup norm. Let $B = \{f \in V \mid \|f\|_{L^\infty} \leq 1\}$ be the closed unit ball in V. Show that there exists a bounded linear functional $\Lambda : V \to \mathbb{R}$ such that $\Lambda(B)$ is an open subset of $\mathbb{R}$.

Proof

$\Lambda = \int_0^1 - \int_{-1}^0$ is bounded, continuous, and maximized by a discontinuous function, so $\Lambda(V)$ is open.

Problem 5

Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded and measurable satisfying $f(x+1) = f(x)$ and $f(2x) = f(x)$ for almost every $x \in \mathbb{R}$. Show then that there exists $c \in \mathbb{R}$ such that $f(x) = c$ for ae x.

Proof

The bad set Z for the conditions is measure zero, so when we take all translations by n and scalings by 2^n, the union is also measure zero.

Outside of this set, f must be constant. Let's use boundedness with the Lebesgue differentiation theorem. Fix x_0, y_0 outside of the scalings/translations of Z (which we'll just call Z now). Fix $\varepsilon > 0$ and pick $r > 0$ such that

$$\left| f(x_0) - \fint_{x_0-r}^{x_0+r} f(t)\,dt \right| < \varepsilon$$
$$\left| f(y_0) - \fint_{y_0-r}^{y_0+r} f(t)\,dt \right| < \varepsilon$$

Find $\delta > 0$ such that $|A| < \delta \implies \int_A |f| < \varepsilon r$.

Then translate x_0 by some scalings/translations so that $|x_0 - y_0| < \delta/2$. Then apply the triangle inequality to $|f(x_0) - f(y_0)|$ and note that the overlap between the two integrals cancels except on a set of measure at most δ, and we're done.

Another Perspective

This problem asks to show that the only $\times 2$-invariant function on the unit circle is a constant almost everywhere.

This follows by ergodicity of $(\mathbb{T}, m, \times 2)$.

Another Proof (With Fourier Series)

Discarding null sets, assume $f : \mathbb{T} \to \mathbb{R}$ satisfies $f(2x) = f(x)$, where x is taken modulo 1. Since f is bounded and measurable, we can describe its Fourier coefficients (it's in every L^p space):

$$\begin{aligned}
\hat{f}(k) &= \int_0^1 f(x)e^{-2\pi ikx}\,dx \\
&= \int_0^1 f(2^n x)e^{-2\pi ikx}\,dx \\
&= 2^{-n}\int_0^{2^k} f(y)e^{-2\pi ik2^{-n}y}\,dy \\
&= 2^{-n}\left(\int_0^1 f(y)e^{-2\pi ik2^{-n}y}\,dy + \int_1^2 + \int_2^4 + \dots\right) \\
&= 2^{-n}\sum_{j=0}^{2^k-1}\int_0^1 f(y)e^{-2\pi ik2^{-n}(y+j)}\,dy \\
&= \left(\sum_{j=0}^{2^k-1} e^{-2\pi ik2^{-n}j}\right)2^{-n}\int_0^1 f(y)e^{-2\pi ik2^{-n}y}\,dy
\end{aligned}$$

where the constant in front vanishes if $k2^{-n}$ is not an integer (since then we have a sum of roots of unity).

Thus when n isn't zero, $\hat{f}(n) = 0$. In other words, f has the same Fourier series as a constant.

Problem 6

Let $f \in L^2(\mathbb{C})$. Define

$$g(z) = \int_{|w-z|\leq 1} \frac{|f(w)|}{|z-w|}\,dA(w)$$

where dA is Lebesgue measure. Show then that $|g(z)| < \infty$ ae and $g \in L^2(\mathbb{C})$.

Proof

Let $C = \int_{|u|\le 1} \frac{1}{|u|}\, dA(u) < \infty$. We write

$$\begin{aligned}
|g(z)|^2 &= \left(\int \frac{|f(w)|}{|w-z|}\right)^2 \\
&\le \left(\int \frac{|f(w)|^2}{|w-z|}\right)\left(\int \frac{1}{|w-z|}\right) \\
&< C\int \frac{|f(w)|^2}{|w-z|}
\end{aligned}$$

by Cauchy-Schwarz. We then apply Tonelli to $\int |g|^2$ and get

$$\begin{aligned}
\int |g|^2 &\le C\int_{\mathbb{C}}\int_{|w-z|\le 1} \frac{|f(w)|^2}{|w-z|}\, dA(w)\, dA(z) \\
&\le C\int_{\mathbb{C}} |f(w)|^2 \int_{|z-w|\le 1} \frac{1}{|z-w|}\, dz\, dw \\
&\le C^2 \|f\|^2 < \infty
\end{aligned}$$

Thus g is finite ae and also L^2.

Problem 7

Prove that here exists a meromorphic function f on $\mathbb{C}$ with the following properties

1. $f(z) = 0$ iff $z \in \mathbb{Z}$
2. $f(z) = \infty$ iff $z - 1/3 \in \mathbb{Z}$
3. $|f(x+iy)| \le 1$ for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}$ with $|y| \ge 1$.

Proof

Let's just go for it.

$$f(z) = \frac{\sin(\pi z)}{\sin(\pi(z - 1/3))}$$

doesn't work, but $f/2$ does. We estimate

$$\begin{aligned}|f(x+y)| &= \left|\frac{e^{i\pi z}-e^{-i\pi z}}{e^{i\pi(z-1/3)}-e^{-i\pi(z-1/3)}}\right| \\ &\le \frac{\left|e^{i\pi z}\right|+\left|e^{-i\pi z}\right|}{\left|\left|e^{i\pi(z-1/3)}\right|-\left|e^{-i\pi(z-1/3)}\right|\right|} \\ &= \frac{e^{-\pi y}+e^{\pi y}}{\left|e^{-\pi y}-e^{\pi y}\right|}\end{aligned}$$

We verify that for $|y| \ge 1$, the above quantity is bounded by 2. Note that it's symmetric in y and so assume $y \ge 1$. Set $x = e^{\pi y}$. Then we need to verify that

$$x + 1/x \le 2\,(x - 1/x)$$

or equivalently

$$\begin{aligned}\frac{x^2+1}{x} &\le \frac{2x^2-1}{x} \\ \frac{3}{x} &\le 2\end{aligned}$$

and so if $x \ge 3/2$, we're done. We have $x = e^{\pi y}$ which is at least $e^{\pi} > 3/2$ for $y \ge 1$.

Problem 8

Show that a harmonic function $u : \mathbb{D} \to \mathbb{R}$ is uniformly continuous if and only if it admits the representation

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) f(e^{i\theta})\,d\theta$$

with $f : \partial\mathbb{D} \to \mathbb{R}$ continuous.

Proof

It is a standard fact that u is uniformly continuous if and only if it admits a continuous extension to $\partial\mathbb{D}$.

If u has a continuous extension to $\partial\mathbb{D}$, just apply the Poisson integral formula.

(To prove it, apply the mean value formula after a conformal map $w \mapsto \frac{w+z}{1+\bar{z}w}$ and simplify the change of variables.)

Conversely, suppose u has the above representation. Need to show that u extends to f continuously.

Fix $e^{i\theta_0} \in \partial\mathbb{D}$. Fix $\varepsilon > 0$. Pick δ_1 such that $|\theta - \theta_0| < \delta$ implies $\left|f(e^{i\theta_0}) - f(e^{i\theta})\right| < \varepsilon$ by continuity of f.

Since $\partial\mathbb{D}$ is compact, let $M = \max\left|f(e^{i\theta})\right|$. Now finally pick $\delta > 0$ small enough that

$$\left|z - e^{i\theta}\right| < \delta$$

and $|\theta - \theta_0| \geq \delta_1$ implies that Re(stuff) $< \varepsilon/2M$.

Note that $\int_0^{2\pi}$ Re(stuff) $= 2\pi$, and so do a classic "add in an integral" estimate

$$\left|u(z) - f(e^{i\theta_0})\right| = \frac{1}{2\pi}\left|\int \mathrm{Re(stuff)}\left(f(e^{i\theta}) - f(e^{i\theta_0})\right)\, d\theta\right|$$

Near θ_0, use that f gets small. Away from θ_0, use that the real part of stuff gets small.

Problem 9

Consider $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ with the following properties.

1. For each z, the map $w \mapsto F(z, w)$ is injective
2. For each w, the map $z \mapsto F(z, w)$ is holomorphic
3. $F(0, w) = w$ for $w \in \mathbb{C}$.

Show then that $F(z, w) = a(z)w + b(z)$ where a and b are entire functions with $a(0) = 1$, $b(0) = 0$ and $a(z) \neq 0$ for $z \in \mathbb{C}$.

Proof

Let's define

$$G(z, w) = \frac{F(z, w) - F(z, 0)}{F(z, 1) - F(z, 0)}$$

and try to show that $G(z, w) = w$. Then $a(z) = F(z, 1) - F(z, 0)$ and $b = F(z, 0)$.

Trivially for $w = 1$ and $w = 0$, we get that $G = w$. Otherwise, note that the denominator of G is never zero by injectivity, and is holomorphic in z. For $w \neq 1, 0$, we get that $G(z, w)$ is entire and misses 0 and 1, so is constant by Picard's little theorem.

Note that $G(0, w) = w$, so $G(z, w) = w$ for all z.

Problem 10

Let f_n be holomorphic functions on $\mathbb{D}$ with

$$F(z) := \sum |f_n(z)|^2 \leq 1$$

for all z. Show that the series defining F converges uniformly on compact subsets of $\mathbb{D}$ and that F is subharmonic.

Proof

Since f_n is holomorphic, $|f_n|^2$ is subharmonic. We thus have a monotone increasing limit of subharmonic functions (the partial sums).

Let's try to repeat the proof of Harnack's Principle for subharmonic functions instead of harmonic (which is false in general), but here with F bounded and some other neat facts, it'll be true here that the limit is subharmonic and the convergence is locally uniform.

If partial sums $g_n \to F$ locally uniformly, then g_n's continuity tells us that F is also, so sub-mean-value property is easy to verify by the monotone convergence theorem (or uniform convergence on compact sets).

$$F(z_0) = \lim g_n(z_0) = \lim \fint_{\partial B(z_0,r)} g_n = \fint F$$

Now for local uniform convergence. Fix $K \subseteq \mathbb{D}$ and $\varepsilon > 0$. Pick $r > 0$ buffer room. Cover K by balls of radus $r/2$. The average integral of $F - g_n$ over these balls goes to zero as $n \to \infty$ (using that F is bounded and monotone convergence theorem). Pick N large so that it's always less than ε.

Then $g_M - g_N$ is positive subharmonic so it satisfies the sub-Poisson-integral-formula ($\leq$ instead of $=$) which gives us a bound on $g_M - g_N$ on any point in the interior of any of these balls.

This bound tells us that g_n is uniformly Cauchy and we're done.

Problem 11

Let $f : \mathbb{D} \to \mathbb{C}$ be injective and holomorphic with $f(0) = 0$ and $f'(0) = 1$. Show that $\inf\{|w| \mid w \notin f(\mathbb{D})\} \leq 1$ with equality iff $f(z) = z$ for all $z \in \mathbb{D}$.

Proof

If $\inf\{|w| \mid w \notin f(\mathbb{D})\} \geq 1$, then $\mathbb{D} \subseteq f(\mathbb{D})$ and we have an inverse $g : \mathbb{D} \to \mathbb{D}$ which satisfies $g(z) = z$ by the Schwarz lemma.

Problem 12

Let f, g, h be complex-valued functions with $f = g \circ h$. Show that if h is continuous and both f, g are non-constant holomorphic, then h is holomorphic too.

Proof

Let B be the set of bad points z for which $g'(h(z)) = 0$. Outside of B, we can find an analyic local inverse and write $h = g_{\text{loc}}^{-1} \circ f$, so h is analytic outside of B.

Since g is non-constant, $g'(z) = 0$ only on a discrete set. Let Z be the zeroes of g'. Let's show that $h^{-1}(Z)$ is disrete.

Suppose $z_n \to z$ with $h(z_n) \in Z$. If $\{h(z_n)\}$ is infinite, it goes to infinity, but $z_n \to z$, so $h(z_n) \to h(z) < \infty$. Contradiction!

Thus $\{h(z_n)\}$ is finite, so there's a constant subsequence. Thus h is constant on some $z_{n_k} \to z$. This means f is constant here too, so f is a constant everywhere. This is a contradiction!

Thus $h^{-1}(Z)$ is discrete, so h is analytic everywhere outside a discrete set, where it's continuous. Thus h is analytic by the removable singularities result.

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Problem 1

Suppose $f \in L^1$ satisfies

$$\limsup_{h \to 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dx = 0.$$

Show that $f = 0$ almost everywhere.

Proof

Note that the sets $[x, x+h]$ as $h \to 0$ are nicely shrinking about the point x. Thus, since $f \in L^1_{\text{loc}}$ at least, we may apply Lebesgue differentiation and obtain that for almost every $x \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{1}{h} \int f(x+h) - f(x)\, dx = f(x)$$

Note that for every Lebesgue point,

$$|f(x)| \le \limsup_{h\to 0} \int \left| \frac{f(x+h) - f(x)}{h} \right| dx = 0$$

and so $f = 0$ at every Lebesgue point (i.e., a.e.).

Problem 2

Given $f \in L^2(\mathbb{R})$ and $h > 0$, we define

$$Q(f, h) = \int_{\mathbb{R}} \frac{2f(x) - f(x+h) - f(x-h)}{h^2} f(x)\, dx$$

1. Show that $Q(f, h) \ge 0$ for all $f \in L^2$ and all $h > 0$
2. Show that the set

   $$E = \{f \in L^2(\mathbb{R}) \mid \limsup_{h\to 0} Q(f, h) \le 1\}$$

   is closed in $L^2(\mathbb{R})$

Proof

- Part 1

  We need to show that $\int 2f(x)f(x) \ge \int f(x)(f(x+h) - f(x-h))$. This is just Cauchy Schwarz:

  $$\begin{aligned} \int f(x)\,(f(x+h) - f(x-h)) &\le \|f\| \|f(x+h) - f(x-h)\| \\ &\le \|f\|\,(\|f(x+h)\| + \|f(x-h)\|) \\ &= 2\|f\|^2 \end{aligned}$$

  We are assuming there that f is real valued.

- Part 2

  We will show closedness directly and be reminded that Fourier transforms of translations are easy to calculate. Since $Q(f, h)$ is really an inner product, we will rewrite it with Plancherel.

Let f_n satisfy $\limsup_{h\to 0} Q(f_n, h) \le 1$ and $f_n \to f$ in L^2. We can write

$$\begin{aligned}
Q(f,h) &= \int_{\mathbb{R}} \frac{2f(x) - f(x+h) - f(x-h)}{h^2} f(x)\,dx \\
&= \int_{\mathbb{R}} \widehat{\left(\frac{2f(x) - f(x+h) - f(x-h)}{h^2}\right)}(t)\widehat{f(x)}(t)\,dt \\
&= \int_{\mathbb{R}} \frac{1}{h^2}\left(2\widehat{f}(t) - e^{ith}\widehat{f}(t) - e^{-ith}\widehat{f}(t)\right)\widehat{f}(t)\,dt \\
&= \int_{\mathbb{R}} \frac{2-2\cos(th)}{h^2}\left|\widehat{f}(t)\right|^2 dt.
\end{aligned}$$

At this point, it's now easy to see why Part 1 holds, since $\cos \le 1$.

Furthermore, we should see that the multiplier $\frac{2-2\cos(th)}{h^2}$ does have a limit as $h \to 0$. In fact, L'Hôpital's rule implies the limit is t^2.

First, a sketch of our goals:

We would like to apply the dominated convergence theorem to take a limit in h inside. However, we need to know that each $Q(f,h)$ is finite. Since cos is roughly quadratic at the origin, we'll try to bound this by something like $\int t^2|\widehat{f}(t)|^2$ (the expected limit).

If $\frac{2-2\cos(th)}{h^2}|\widehat{f}(t)|^2$ is dominated, then by taking a limit we get $t^2|\widehat{f}(t)|^2$. Our goal is that this should have integral bounded by 1. This is an application of Fatou to compare with $\int t^2|\widehat{f_n}(t)|^2$. This itself will be small by Fatou in h. Let's start.

First,

$$\begin{aligned}
1 &\ge \limsup_{h\to\infty} \int_{\mathbb{R}} \frac{2-2\cos(th)}{h^2}|\widehat{f_n}(t)|^2\,dt \\
&\ge \liminf \int \frac{2-2\cos(th)}{h^2}|\widehat{f_n}(t)|^2\,dt \\
&\ge \int t^2|\widehat{f_n}(t)|^2\,dt
\end{aligned}$$

by Fatou in h.

To apply Fatou in n, take subsequences such that $f_n \to f$ ae and $\hat{f}_n \to \hat{f}$ ae. Then by Fatou again,

$$1 \ge \liminf_n \int t^2|\widehat{f_n}(t)|^2\,dt \ge \int t^2|\widehat{f}(t)|^2\,dt$$

Now the only remaining question is if $\frac{2-2\cos(th)}{h^2}|\widehat{f}(t)|^2$ is dominated. Note that be rewriting, we get

$$t^2\left(\frac{2-2\cos(th)}{t^2h^2}\right)$$

where the second term is bounded (since it converges to 1 at the origin and goes to 0 at infinity). Let's say it's bounded by C.

Then the term in question is dominated by $Ct^2|\widehat{f}(t)|^2$, which we showed is integrable. Thus

$$\begin{aligned}\lim_{h\to\infty}\int_{\mathbb{R}}\frac{2-2\cos(th)}{h^2}|\widehat{f}(t)|\,dt &= \int\lim\frac{2-2\cos(th)}{h^2}|\widehat{f}(t)| \\ &= \int t^2|\widehat{f}(t)|^2\,dt \\ &\le \liminf_n\int t^2|\widehat{f_n}(t)|^2\,dt \le 1\end{aligned}$$

Problem 3

Suppose $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|f(x)f(y)|}{|x-y|^2+\varepsilon^2}\,dx\,dy < \infty.$$

Show that $f = 0$ almost everywhere.

Proof

Our first thoughts should be Lebesgue differentiation and if we can get rid of that ε.

Note that as $\varepsilon \to 0$, the integrand only increases. If we had a limit instead of limsup, we could apply the Monotone Convergence Theorem. Consider any sequence $\varepsilon_n \to 0$ such that $\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|f(x)f(y)|}{|x-y|^2+\varepsilon_n^2}$ goes to the limsup.

On this sequence we have a limit, and thus apply Monotone Convergence (secretly rewriting everything with Tonelli as an integral over $\mathbb{R}^2$) to get:

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|f(x)f(y)|}{|x-y|^2}\,dx\,dy < \infty$$

Now let's apply Lebesgue differentiation. Let a be a Lebesgue point. Then

$$\begin{aligned}\infty > \int_{\mathbb{R}^2} \frac{|f(x)f(y)|}{|x-y|^2}\,dx\,dy &\geq \int_{a-r}^{a+r}\int_{a-r}^{a+r} \frac{|f(x)f(y)|}{|x-y|^2}\,dx\,dy \\ &\geq \int_{a-r}^{a+r}\int_{a-r}^{a+r} \frac{|f(x)f(y)|}{4r^2}\,dx\,dy \\ &= \left(\int_{a-r}^{a+r} \frac{|f(x)|}{2r}\,dx\right)^2 \to |f(a)|^2\end{aligned}$$

by Fubini–Tonelli and in the last step, Lebesgue differentiation.

Note that $[a-r, a+r]^2$ shrinks to a point as $r \to \infty$, so (technically by Lebesgue dominated convergence) we have that

$$\int_{a-r}^{a+r}\int_{a-r}^{a+r} \frac{|f(x)f(y)|}{|x-y|^2}\,dx\,dy \to 0$$

as $r \to 0$.

Thus $f = 0$ at all Lebesgue points.

Problem 4

1. Fix $1 < p < \infty$. Show that

$$f \mapsto [Mf](x,y) = \sup_{r>0,\rho>0} \frac{1}{4r\rho}\int_{-r}^{r}\int_{-\rho}^{\rho} f(x+h, y+\ell)\,dh\,d\ell$$

   is bounded on $L^p(\mathbb{R}^2)$.

2. Show that

$$[A_r f](x,y) = \frac{1}{4r^3}\int_{-r}^{r}\int_{-r^2}^{r^2} f(x+h, y+\ell)\,dh\,d\ell$$

   converges to f ae in the plane as $r \to 0$.

Proof

This looks like Lebesgue differentation, except the aspect ratio of the ball (or rather, rectangle) can vary dramatically. The usual version we have of the L^p boundedness of the maximal function deals only with normal balls, and the version of Lebesgue differentiation we have can only deal with sets that "shrink nicely" (must have significant mass and be contained in balls). But these $[-r, r]\times [-r^2, r^2]$ sets do not shrink nicely.

We might hope to show a weak L^1 bound and use the trivial L^∞ bound to then apply Marcinkiewicz to imply an L^p bound. However, this has little hope of happening.

For the weak L^1 bound, we need to take a collection of rectangles on which Mf is witnessed to be large and somehow find a disjoint subfamily whose enlargements cover the original set.

But imagine the union of $[0, x] \times [0, 1/x]$. There's no hope here of finding a disjoint subfamily whose enlargements cover the set, since the aspect ratio diverges.

Thus instead let's apply the L^p boundedness of the usual maximal function twice with Fubini.

- Part 1

  Let G be the usual maximal function (since the problem takes M already). Suppose $f \in L^p$. Let $f_x(y) = f(x, y)$. Since

  $$\infty > \int\int |f_x(y)|^p \, dx \, dy,$$

  we must have that $f_x \in L^p$ for ae x. In particular by L^p boundedness of the Hardy-Littlewood Maximal function, $\|Gf_x\|_{L^p} \le \|f_x\|_{L^p}$.

  Note that

  $$\begin{aligned} Mf(x, y) &= \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\rho>0} \frac{1}{2\rho} f_{x+h}(y + \ell) \, dh \right) d\ell \\ &= G_x \left(G_y f_x \right) \end{aligned}$$

  where the subscripts on G are just for clarity.

  Thus,

  $$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} \left|G_x(G_y f_x)\right|^p dx \, dy &\le \int_{\mathbb{R}} \int_{\mathbb{R}} \left|G_y f_x\right|^p dx \, dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left|G_y f_x\right|^p dy \, dx \\ &\le \int_{\mathbb{R}} \int_{\mathbb{R}} |f_x|^p \, dy \, dx = \iint |f|^p \end{aligned}$$

  and thus M is bounded in L^p.

- Part 2

  We have L^p boundedness of M, so let's mimic the proof of Lebesgue differentiation. We compare to a continuous function. (Also, we're assuming $f \in L^p$.)

  Let $g \in C_c(\mathbb{R}^2)$. Our goal is to show that $A_r f - f \to 0$ ae. We write

  $$\begin{aligned}(A_r f - f)(x,y) &= \frac{1}{4r^3} \int_{-r}^{r} \int_{-r^2}^{r^2} f(x+h, y+\ell) - f(x,y)\, dh\, d\ell \\ &= \fint_{-r}^{r} \fint_{-r^2}^{r^2} f(x+h, y+\ell) - g(x+h, y+\ell)\, dh\, d\ell \\ &\quad + \fint_{-r}^{r} \fint_{-r^2}^{r^2} g(x+h, y+\ell) - g(x,y)\, dh\, d\ell \\ &\quad + \fint_{-r}^{r} \fint_{-r^2}^{r^2} g(x,y) - f(x,y)\, dh\, d\ell\end{aligned}$$

  Taking absolute values and a limsup in r, we get

  $$\limsup_{r\to 0} |A_r f - f| \leq M(f-g)(x,y) + 0 + |g-f|(x,y)$$

  If $\limsup_{r\to 0} |A_r f - f| > 2\alpha$, then either $M(f-g) > \alpha$ or $|f-g| > \alpha$. But,

  $$m\{M(f-g) > \alpha\} \leq \frac{1}{\alpha^p} \|M(f-g)\|_{L^p} \leq \frac{C}{\alpha^p} \|f-g\|_{L^p}$$

  and

  $$m\{|f-g| > \alpha\} \leq \frac{1}{\alpha^p} \|f-g\|_{L^p}$$

  Taking $\|f-g\|_{L^p}$ small enough that both of the above terms are bounded by some $\varepsilon/2 > 0$, we get that $m\{\limsup |A_r f - f| > 2\alpha\} \leq \varepsilon$ for all ε, and so $\limsup |A_r f - f| = 0$ almost everywhere, as desired.

Problem 5

Let μ be a real-valued Borel measure on $[0,1]$ such that

$$\int_0^1 \frac{1}{x+t}\, d\mu(t) = 0 \text{ for all } x > 1.$$

Show that $\mu = 0$.

Proof

Hopefully the linear span of the functions $t \mapsto \frac{1}{x+t}$ is dense in $C([0,1])$. If so, then Riesz Representation finishes the proof.

Recall Stone-Weierstrass: if an sub-algebra of $C(X)$ (for X compact) contains the constants and separates points, then it's dense.

There's another useful version: if X is a locally compact Hausdorff space, then a sub-algebra which vanishes nowhere and separates points is dense. Recall that vanishing nowhere means for every point, there's a function non-zero at it.

Let $S = \operatorname{span}\{t \mapsto \frac{1}{x+t} \mid x > 1\}$. I claim this is a sub-algebra.

We need only verify that we can solve

$$\frac{1}{x+t} \cdot \frac{1}{y+t} = \frac{A}{x+t} + \frac{B}{y+t}$$

but this is easy by partial fractions. We get $-A = B = \frac{1}{x-y}$.

Next, we need to check that this sub-algebra separates points, but this is trivial.

Next, S is clearly non-vanishing. If we needed to show it contains the constants instead, we could just take $x/(x+t) \in S$. By sending $x \to \infty$, we get that $1 \in \overline{S}$.

Thus $\overline{S} = C([0,1])$.

First, I claim that μ is a finite measure. This is because its integral against $1/(x+t)$ is finite and $1/(x+t) \geq c$.

Thus the Riesz representation theorem says that since μ is zero on a set of functions whose closure is $C([0,1])$ (and hence μ is zero tested against all continuous functions), in fact $\mu = 0$.

Problem 6

Let $\mathbb{T}$ denote the unit circle in the complex plane and let $\mathscr{P}(\mathbb{T})$ denote the space of Borel probability measure on $\mathbb{T}$. Similarly, define $\mathscr{P}(\mathbb{T} \times \mathbb{T})$. Fix $\mu, \nu \in \mathscr{P}(\mathbb{T})$ and define

$$\mathscr{M} = \left\{ \gamma \in \mathscr{P}(T \times T) \mid \iint_{\mathbb{T}\times\mathbb{T}} f(x) g(y)\, d\gamma(x,y) = \int_{\mathbb{T}} f(x)\, d\mu(x) \cdot \int_{\mathbb{T}} g(y)\, d\nu(y) \text{ for all } f, g \in C(\mathbb{T}) \right\}.$$

Show that $F : \mathscr{M} \to \mathbb{R}$ defined by

$$F(\gamma) = \iint_{\mathbb{T}^2} \sin^2\left(\frac{\theta - \phi}{2}\right) d\gamma(e^{i\theta}, e^{i\phi})$$

achieves its minimum on $\mathscr{M}$.

Proof

- Trick method:

  "Recall" that $\sin^2(\theta/2) = \frac{1}{2}(1 - \cos(\theta))$.

  Thus,

  $$\begin{aligned}\sin^2\left(\frac{\theta - \phi}{2}\right) &= \frac{1}{2}(1 - \cos(\theta - \phi)) \\ &= \frac{1}{2}(1 - \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi))\end{aligned}$$

  whcih is a sub of 3 functions of the form $f(\theta)g(\phi)$. Thus F is constant on M, being independent of γ.

- Non-trick method:

  The basic idea is as follows: take an infimizing sequence and show a subsequential limit (which exists by Banach-Alaoglu) produces the minimum.

  Let's take γ_n such that $F(\gamma_n) \to I$, the infimum.

  Since $\mathscr{P}(\mathbb{T}^2)$ is weakly-star compact by Banach-Alaoglu, being a subset of the unit ball in $C(\mathbb{T}^2)^*$, we make take a weakly-star convergent subsequence $\gamma_{n_k} \to \gamma$. Technically here we use sequential-weak*-compactness, by the separability of $C(\mathbb{T}^2)$.

  First, we need to check that γ is a probability measure. But this is trivial by weak* limits and integrating against 1.

  Next,

  $$\iint f(x)g(y)\,d\gamma = \lim \iint fg\,d\gamma_{n_k} = \int f\,d\mu \int g\,d\nu$$

  so $\gamma \in \mathscr{M}$.

  Finally, $F(\gamma) = \int \sin^2\left(\frac{\theta-\phi}{2}\right)\,d\gamma = \lim F(\gamma_{n_k}) = I$.

  Technically, we might want to show that γ is real-valued. This follows by the portmanteau theorem and noting that each γ_{n_k} is real-valued.

Problem 7

Let $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be (jointly) continuous and holomorphic in each variable separately. Show that $z \mapsto F(z, z)$ is holomorphic.

Proof

First, since $F(a,b)$ is holomorphic, we have

$$F(a,b) = \frac{1}{2\pi i}\int_{|\zeta-b|=r}\frac{F(a,\zeta)}{\zeta-b}\,d\zeta$$

Next,

$$F(a,\zeta) = \frac{1}{2\pi i}\int_{|\xi-a|=r}\frac{F(\xi,\zeta)}{\xi-a}\,d\xi$$

Putting this together, we get

$$F(a,b) = \frac{1}{(2\pi i)^2}\int_{|\zeta-b|=r}\frac{1}{\zeta-b}\int_{|\xi-a|=r}\frac{F(\xi,\zeta)}{\xi-a}\,d\xi\,d\zeta$$

If we can apply Fubini-Tonelli, we can rewrite this as a single integral. Luckily, F is continuous, so its integral on compact sets is finite, and hence $\frac{F(\xi,\zeta)}{(\xi-a)(\zeta-b)}$ is integrable. Thus we write

$$F(a,b) = \frac{1}{(2\pi i)^2}\iint_{|\zeta-b|=r,|\xi-a|=r}\frac{F(\xi,\zeta)}{(\xi-a)(\zeta-b)}\,d\xi\,d\zeta$$

Now

$$f(z) = F(z,z) = \frac{1}{(2\pi i)^2}\iint_{|\zeta-z|=r,|\xi-z|=r}\frac{F(\xi,\zeta)}{(\xi-z)(\zeta-z)}\,d\xi\,d\zeta$$

I claim we can differentiate under the integral sign. The integrand is continuous, the domain is compact, and the integrand is smooth in z. There are no problems applying Lebesgue Dominated Convergence.

Thus f is holomorphic.

Problem 8

Determine the supremum of

$$\left|\frac{\partial u}{\partial x}(0,0)\right|$$

among all harmonic functions $u : \mathbb{D} \to [0,1]$ where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Prove that your answer is correct.

Proof

Since $\mathbb{D}$ is simply connected, all such harmonic functions are real parts of holomorphic functions. Thus we consider $S = [0,1] \times i\mathbb{R}$, the strip and all functions $f : \mathbb{D} \to S$.

Let $f : \mathbb{D} \to S$. Without loss of generality, we can assume $f(0) \in \mathbb{R}$, since translating won't affect any derivatives.

Then we need to bound $\mathrm{Re}(f'(0)) = \frac{\partial u}{\partial x}$. But up to precomposition with rotation, we can assume that $f'(0)$ is real. In other words, we need to bound $|f'(0)|$.

Let's consider the following conformal map $T : S \to \mathbb{D}$ via

$$T(z) = \frac{\exp(i\pi z) - i}{\exp(i\pi z) + i}$$

which sends S to the upper half disc, then to $\mathbb{D}$ itself.

Thus we are looking for the maximum of f'. Note that $T \circ f : \mathbb{D} \to \mathbb{D}$. Finally, we need to make sure $T \circ f(0) = 0$. This might not be true. Let $T(f(0)) = \alpha$. Let $\psi(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ be the automorphism of $\mathbb{D}$ sending α to 0.

Then $\psi \circ T \circ f : \mathbb{D} \to \mathbb{D}$ and sends 0 to 0. Thus $|(\psi \circ T \circ f)'(0)| \le 1$ by the Schwarz lemma.

We calculate

$$\begin{aligned} 1 \ge |(\psi \circ T \circ f)'(0)| &= |\psi'(\alpha)||T'(f(0))||f'(0)| \\ &= \frac{1}{1-|a|^2} \frac{2\pi}{\left|e^{i\pi f(0)} + i\right|^2} |f'(0)| \\ &\ge \frac{\pi}{2}|f'(0)| \end{aligned}$$

since $e^{i\pi f(0)}$ is on the unit circle, since $f(0)$ is real.

Thus $|f'(0)| \le 2/\pi$. Furthermore, this is realized when $f(0) = 1/2$, since then $T(1/2) = 0$, so $\psi = \mathrm{id}$ and $|f'(0)| = 2/\pi$.

Take $f(z) = T^{-1}(z)$. This is well-defined and satisfies the above.

Problem 9

Consider the formal product

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^z \left(1 - \frac{z}{n}\right)$$

1. Show that the product converges for any $z \in (-\infty, 0)$

2. Show that the resulting function extends from this interval to an entire function of $z \in \mathbb{C}$.

Proof

- Part 1

  The Bernoulli inequality states that $(1+x)^r \geq 1 + rx$ for every real $x \geq 1$ and $x \geq -1$.

  Thus for $z < 0$, we have $(1 - z\frac{1}{n}) < (1 + \frac{1}{n})^{-z}$. In particular, the terms in the product are all bounded by 1 (and all real).

  Thus the product converges.

- Part 2

  Recall the binomial *series*:

  $$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

  where $\alpha \in \mathbb{C}$ and $\binom{\cdot}{k}$ is as usual (generalized slightly).

  Recall also that the product $\prod(1 + a_i)$ converges iff $\sum |a_i|$ converges.

  We then get

  $$\left(1 + \frac{1}{n}\right)^z = 1 + \frac{z}{n} + \frac{z(z-1)}{2n^2} + \cdots = 1 + \frac{z}{n} + O(n^{-2})$$

  We now consider

  $$\sum_{n=2}^{\infty} \left| \left(1 + \frac{1}{n}\right)^z \left(1 - \frac{z}{n}\right) - 1 \right| \leq \sum \left| 1 - \frac{z^2}{n^2} + O(n^{-2}) - 1 \right| < \infty.$$

  For z bounded, the sum converges at a uniform rate. A uniform rate probably implies uniform convergence of the product.

Problem 10

Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and $\Omega = \mathbb{C}^* \setminus \{0, 1\}$. Let $f : \Omega \to \Omega$ be holomorphic.

1. Prove that if f is injective, then $f(\Omega) = \Omega$.
2. Make a list of all such injective functions f.

Proof

Suppose f is injective and $f(\Omega) \neq \Omega$.

Without loss of generality, let's consider $U = \mathbb{C}^* \setminus \{0, \infty\}$ and consider functions on U instead.

Suppose $f : U \to U$ is injective. If f has an essential singularity at the origin, it takes on all values (with one exception) infinitely often by the big Picard theorem. This can't happen, so f doesn't have an essential singularity at the origin.

At the origin, let's consider the behavior of f. If f has a removable singularity, then we extend to $f : \mathbb{C} \to \mathbb{C}$, so $f(z) = az + b$.

Otherwise, suppose f has a pole of order at least 2. Then for γ a small circle about the origin, note that $1/f$ has a zero of order at least 2, so $1/f(\gamma)$ winds about the origin twice. It also winds about points near the origin twice, and so $1/f$ attains those values twice as well. Thus f is not injective, a contradiction.

Therefore $f(z) = az+b+c/z$ for some a, b, c. Note that if $a, c \neq 0$, then there are other zeroes, a contradiction. Similarly, we need $b = 0$. Thus $f(z) = az$ or $f(z) = a/z$. Note that both of these are surjective.

We translate back to Ω to get

$$f(z) = 1 + \frac{w-1}{(a-1)w+1}$$

or

$$f(z) = 1 + \frac{w}{(a-1)w-a}$$

for some $a \neq 0$.

Problem 11

For $R > 1$, let A_R be the annulus $\{1 < |z| < R\}$. Assume there is a conformal mapping $F : A_{R_1} \to A_{R_2}$. Prove that $R_1 = R_2$.

Proof

- Slightly sketchy:

  By Schwarz reflection, we may reflect F about the unit circle, obtaining a new mapping $F : \{1/R_1 < |z| < R_1\} \to \{1/R_2 < |z| < R_2\}$.

  We repeatedly expend F by reflecting the annulus across its inner border, eventually obtaining $F : B(0, R_1) \to B(0, R_2)$ (removing the obviously removable singularity) and furthermore, $F : B(0, 1) \to B(0, 1)$.

Thus F is a rotation, since $F(0) = 0$. In particular, $B(0, R_1) = B(0, R_2)$, so $R_1 = R_2$.

- Less sketchy:

  We know that F maps the boundary to the boundary, so by composing with an inversion if necessary, F sends the unit circle to itself.

  Next, note that $\log|f(z)|$ is well-defined and harmonic on A_{R_1}, since f avoids zero. Furthermore, $\log(R_2 - R_1)\log|z|$ is also harmonic and has the same boundary values. Since the Dirichlet problem can be solved on the annulus, these functions are equal.

  $$\log|f(z)| = \log(R_2 - R_1)\log|z|$$

  Thus $|f(z)| = |z|^{\log(R_2 - R_1)}$. These functions (without $|\cdot|$ are analytic in the slit annulus, so they're equal up to rotation, but f analytically continues to the annulus, so $z^{\log(R_2-R_1)}$ must too.

  Thus $\log(R_2 - R_1)$ is a positive integer, and even more, it must be 1, since otherwise f is not injective.

  Thus $R_2 - R_1 = 0$.

Problem 12

Let $f(z)$ be bounded and holomorphic on the unit disc $\mathbb{D}$. Prove that for any $w \in \mathbb{D}$, we have

$$f(w) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(z)}{(1 - \bar{z}w)^2}\, dA(z)$$

where $dA(z)$ is integration with respect to Lebesgue measure.

This is Bergman's kernel formula.

Proof

- Weird Method:

  Let $F : A^2 \to A^2$ via $F(f) = f(w)$. This is a bounded linear functional on A^2, the space of holomorphic functions in $L^2(\mathbb{D})$.

  Thus by the Riesz Representation theorem, there's an element of A^2 called g_w such that $F(f) = \langle f, g_w \rangle$. It suffices to show that $\overline{g_w}(z) = \frac{1}{\pi} \cdot \frac{1}{(1-\bar{z}w)^2}$.

To calculate $g_w(z)$, we write

$$
\begin{aligned}
g_w(z) = \langle g_w, g_z \rangle &= \sum_{n=0}^{\infty} \langle g_w, e_n \rangle \overline{\langle g_z, e_n \rangle} \\
&= \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z) \\
&= \sum_{n=0}^{\infty} \frac{1}{\pi}(n+1)(\overline{w}z)^n = \frac{1}{\pi(1-\overline{w}z)^2}
\end{aligned}
$$

recalling the standard orthonormal basis $e_n = \sqrt{\frac{n+1}{\pi}} z^n$.

- More Reasonable Method:

  If $w = 0$, we're done, so assume $w \neq 0$.

  Let's use the standard differentials: $dz = dx + idy$, $d\bar{z} = dx - idy$. We have that $d\bar{z} \wedge dz = 2idx \wedge dy$. Furthermore,

$$
\begin{aligned}
\frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\
\frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)
\end{aligned}
$$

  The integrand is

$$
\frac{f(z)}{(1-w\bar{z})^2} dx \wedge dy
$$

  The function looks like a derivative $\frac{\partial}{\partial \bar{z}}$. Write

$$
\frac{\partial}{\partial \bar{z}}\left(\frac{f(z)}{w(1-w\bar{z})}\right) = \frac{f(z)}{(1-w\bar{z})^2}
$$

  since $\frac{\partial f}{\partial \bar{z}} = 0$ because f is analytic.

  Then if we write

$$
F = \frac{f(z)}{w(1-w\bar{z})} dz
$$

then

$$
\begin{aligned}
dF &= \frac{f(z)}{(1-w\bar{z})^2} d\bar{z} \wedge dz \\
&= 2i \frac{f(z)}{(1-w\bar{z})^2}
\end{aligned}
$$

Thus we can apply Stokes' Theorem!

$$
\begin{aligned}
\frac{1}{\pi} \int_{\mathbb{D}} \frac{f(z)}{(1-w\bar{z})^2} dx \wedge dy &= \frac{1}{2\pi i} \int_{\mathbb{D}} dF \\
&= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} F \\
&= \frac{1}{2\pi i w} \int_{\partial\mathbb{D}} \frac{f(z)}{1-w\bar{z}}\, dz \\
&= \frac{1}{2\pi i w} \int_{\partial\mathbb{D}} \frac{z f(z)}{z-w}\, dz \\
&= \frac{2\pi i w f(w)}{2\pi i w} = f(w)
\end{aligned}
$$

- Ahlfors Approach (best!): We write

$$
\begin{aligned}
\text{RHS} &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{f(re^{i\theta})}{(1-re^{-i\theta}w)^2} r\, dr\, d\theta \\
&= \frac{1}{\pi} \int_0^1 r \int_0^{2\pi} \frac{f(re^{i\theta})}{(1-re^{-i\theta}w)^2}\, d\theta\, dr \\
&= \frac{1}{\pi} \int_0^1 r \int_{|z|=r} \frac{f(z)}{(1-r^2w/z)^2} \frac{-i}{z}\, dz\, dr
\end{aligned}
$$

There is a double pole at $z = r^2 w$ (where $\left|r^2 w\right| < r$, so it's inside the circle $|z| = r$). There is also an apparent pole at $z = 0$, but this is illusory (it is in fact a removable singularity).

We calculate the residues:

$$
\begin{aligned}
\operatorname{Res}_{r^2w}\left(\frac{f(z)}{(1-r^2w/z)^2} \frac{1}{zi}\right) &= \operatorname{Res}_{r^2w}\left(\frac{f(z)z}{i(z-r^2w)^2}\right) \\
&= \frac{1}{i}\left(f(z)z\right)'(r^2w) = \frac{r^2w}{i} f'(r^2w) + \frac{1}{i} f(r^2w)
\end{aligned}
$$

We get

$$\begin{aligned}
\text{RHS} &= \frac{1}{\pi} 2\pi \int_0^1 r^3 w f'(r^2 w) + r f(r^2 w)\, dr \\
&= 2 \int_0^1 r^3 w f'(r^2 w) + r f(r^2 w)\, dr \\
&= \int_0^1 \frac{d}{dr} \left(r^2 f(r^2 w) \right) \\
&= f(w)
\end{aligned}$$

- Failed Approach:

  If $w = 0$, we're done by the mean value property.

  Fix $w \neq 0$. Let $\psi : \mathbb{D} \to \mathbb{D}$ be the following automorphism:

  $$\psi(z) = \frac{z - w}{1 - \bar{w} z}$$

  which sends w to 0.

  Define $g(\xi) = f \circ \psi^{-1}(\xi)$. Then $g : \mathbb{D} \to \mathbb{C}$ is still bounded and holomorphic, but now $g(0)$ can be evaluated as follows:

  $$\begin{aligned}
  f(w) = g(0) &= \frac{1}{\pi} \int_{\mathbb{D}} g(\xi)\, dA(\xi) \\
  &= \frac{1}{\pi} \int_{\mathbb{D}} f(\psi^{-1}(\xi))\, dA(\xi) \\
  &= \frac{1}{\pi} \int_{\mathbb{D}} f(z) |\psi'(z)|^2\, dA(z) \\
  &= \frac{1}{\pi} \int_{\mathbb{D}} f(z) \left| \frac{1 - |w|^2}{(1 - \bar{w} z)^2} \right|^2 dA(z)
  \end{aligned}$$

  Note that this integrand is real. Perhaps manipulation can massage this into the desired form, but this seems difficult.

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Problem 1

Let $\{f_n\}$ be a sequence of real-valued Lebesgue measurable functions on $\mathbb{R}$ and let f be another such function. Assume that

1. $f_n \to f$ Lebesgue almost everywhere

2. $\int |x||f_n(x)|\,dx \le 100$ for all n, and

3. $\int |f_n(x)|^2\,dx \le 100$ for all n.

Prove that $f, f_n \in L^1$ and $\|f_n - f\|_{L^1} \to 0$. Also show that neither assummption 2 nor 3 can be omitted while making these deductions.

Proof

To show that $f_n \in L^1$, we note that

$$\begin{aligned}\int |f_n| &= \int_{|x|\le 1} |f_n| + \int_{|x|>1} |f_n| \\ &\le 2^{1/2}\|f\|_{L^2} + \int_{|x|>1} |x||f_n| \le C < \infty.\end{aligned}$$

Here we required both (2) and (3) to ensure that $f_n \in L^1$.

To show that $f \in L^1$, we apply Fatou's lemma to say

$$\int |f| = \int \liminf |f_n| \le \liminf \int |f_n| \le C < \infty.$$

Now to show that $f_n \to f$. We will make estimates on both $|x| < R$ and $|x| > R$.

Note that

$$\int_{|x|>R} |f_n| \le \int_{|x|>R} \frac{|x|}{R}|f_n| \lesssim \frac{1}{R}$$

$$\int_E |f_n| \lesssim |E|^{1/2}$$

when E is any finite measure set. Here we use L^2 boundedness and condition 2 again.

We take R big enough such that $\int_{|x|>R} |f_n| < \varepsilon$ for all n (and also for f. Now by Egorov's theorem, there is a set $E \subseteq [-R, R]$ such that on E, $f_n \to f$ uniformly and such that $|E^c|^{1/2} < \varepsilon$.

Thus we write

$$\begin{aligned}\int |f_n - f| &= \int_{|x|>R} |f_n - f| + \int_E |f_n - f| + \int_{E^c} |f_n - f| \\ &\le 4\varepsilon + \int_E |f_n - f|.\end{aligned}$$

Taking a limsup in n gives us at most 4ε by uniform convergence. Finally, sending $\varepsilon \to 0$ gives us the result.

Problem 2

Let (X, ρ) be a compact metric space with at least two points, and let $C(X)$ be the space of continuous functions $X \to \mathbb{R}$ with the uniform norm. Let D be a dense subset of X and for each $y \in D$, define $f_y \in C(X)$ by $f_y(x) = \rho(x, y)$. Let A be the subalgebra of $C(X)$ generated by the collection $\{f_y \mid y \in D\}$.

1. Prove that A is dense in $C(X)$ under the uniform norm.

2. Prove that $C(X)$ is separable.

Proof

- Part 1

  It's enough to check that A separates points $\left(\forall x \neq y.\ \exists f.\ f(x) \neq f(y)\right)$ and is non-vanishing.

  For separating points, given $x \neq y$, let $f = f_y$. For nonvanishing, given x, let $f = f_y$ for any $y \neq x$.

- Part 2

  Note that X is compact, and hence has a countable dense set. This is an elementary result in topology, but to show this, consider total boundedness of X and note that for every $n \in \mathbb{N}$, there exists $x_1^n, \dots, x_{m_n}^n$ such that for all $x \in X$, there exists $\rho(x_{m_i}^n, x) < \varepsilon$.

  Take $D = \{x_{m_i}^n \mid 1 \leq i \leq n \in \mathbb{N}\}$ as a countable dense subset. Then A is countable and dense in $C(X)$, and hence $C(X)$ is separable.

Problem 3

Let (X, ρ) be a compact metric space and let $P(X)$ be the set of all Borel probability measures on X. Assume $\mu_n \to \mu$ in the weak-star topology on $P(X)$. Prove that $\mu_n(E) \to \mu(E)$ whenever E is a Borel subset of X such that $\mu(\overline{E}) = \mu(E^\circ)$, where $\overline{E}$ is the closure and E° is the interior.

Proof

We apply the portmanteau theorem twice. Since E° is open and $\overline{E}$ is closed, we have

$$\begin{aligned}\mu(E^\circ) &\le \liminf_{n\to\infty} \mu_n(E^\circ) \le \liminf_{n\to\infty} \mu_n(E)\\ &\le \limsup_{n\to\infty} \mu_n(E) \le \limsup_{n\to\infty} \mu_n(\overline{E})\\ &\le \mu(\overline{E})\end{aligned}$$

But $\mu(E^\circ) = \mu(\overline{E}) = \mu(E)$, so the limit exists and is as required.

The parts of the portmanteau theorem we need follow by considering the following.

For an open set U, consider $F = \{f : X \to [0,1] \mid \operatorname{supp} f \subseteq U\}$. Then $\mu(U) = \sup_{f\in F} \int f\, d\mu$.

For any fixed f,

$$\mu_n(U) \ge \int f\, d\mu_n$$

Taking a liminf in f gives us $\liminf \mu_n(U) \ge \int f\, d\mu$ and then a supremum in f gives us the result. For the other result, consider C^c for C closed.

Problem 4

Let $\mathbb{T}$ be the unit circle in the complex plane, and for each $\alpha \in \mathbb{T}$ define the rotation map $R_\alpha : \mathbb{T} \to \mathbb{T}$ by $R_\alpha(z) = \alpha z$. A Borel probability measure μ on $\mathbb{T}$ is called α-invariant if $\mu(R_\alpha(E)) = \mu(E)$ for all Borel sets $E \subseteq \mathbb{T}$.

1. Let m be the Lebesgue measure on $\mathbb{T}$. Show that for every $\alpha \in \mathbb{T}$, m is α-invariant.

2. Prove that if α is not a root of unity, then the set of powers $\{\alpha^n \mid n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$.

3. Prove that if α is not a root of unity, then m is the only α-invariant Borel probablity measure on $\mathbb{T}$.

Proof

- Part 1

  Viewing $\mathbb{T}$ as $[0,1)$, we note that the Lebesgue measure is translation invariant. Note that $\alpha < 1$ under this identification.

For $E \subseteq [0,1)$ with $R_\alpha E \subseteq [0,1)$, translation invariance is all we need. Otherwise, write

$$\begin{aligned} E &= \{x \in E \mid x + \alpha < 1\} \sqcup \{x \in E \mid x + \alpha \geq 1\} \\ &=: E_0 \sqcup E_1 \end{aligned}$$

where $\sqcup$ denotes a disjoint union. Note that $R_\alpha(E_0)$ and $R_\alpha(E_1)$ are also disjoint (by considering them subsets of circle).

We evaluate

$$\begin{aligned} m(R_\alpha(E)) &= m(R_\alpha(E_0 \cup E_1)) \\ &= m(R_\alpha(E_0) \cup R_\alpha(E_1)) \\ &= m(R_\alpha(E_0)) + m(R_\alpha(E_1)) \\ &= m(E_0) + m(E_1) \end{aligned}$$

where in both cases we use invariance of the Lebesgue measure under translations.

- Part 2 Method 1

  Let's enough to show that $\{n\alpha \mid n \geq 0\}$ is dense in $\mathbb{T}$ for α irrational. Let $\varepsilon > 0$.

  The orbit containts countably many distinct points, so by the pigeonhole principle, there are m, n such that $\|n\alpha - m\alpha\|_{\mathbb{T}} < \varepsilon$ (where $\|\cdot\|_{\mathbb{T}}$ denotes "mod 1" distance).

  Thus $\|(n-m)\alpha\| < \varepsilon$, and so $\{k(n-m)\alpha \mid k \geq 0\}$ is no more than ε from any point. We thus have density.

- Part 2 Method 2

  To show $\{n\alpha \mid n \geq 0\}$ is dense in $\mathbb{T}$, we show a stronger result, the equidistribution theorem: for any $0 \leq a < b \leq 1$,

$$\lim_{N \to \infty} \frac{\#\{n \mid a \leq n\alpha \leq b\}}{N} = b - a.$$

  This follows by the pointwise ergodic theorem applied to $\chi_{[a,b]}$ and the point 0, which we'll prove in this case. In general, the pointwise ergodic theorem holds only almost everywhere, but the point 0 in this case does hold.

The hypotheses of the theorem hold because m is rotation invariant and the associated measure preserving system is ergodic (as there are no invariant sets of non-zero or non-full measure).

If the point x is one of the points for which the pointwise ergodic theorem applied to $\chi_{[a,b]}$ holds, then

$$\lim_{N\to\infty} \frac{\#\{n \mid a \le x + n\alpha \le b\}}{N} = b - a$$

and this holds for almost every x. In particular, density is definitely true.

Let's show this version of the pointwise ergodic theorem with the particular point $x = 0$.

For any $f \in L^1(\mathbb{T})$, set

$$A_N f = \frac{1}{N} \sum_{n=0}^{N-1} f(n\alpha) \in \mathbb{R}$$
$$E(f) := \int_{\mathbb{T}} f \, dm$$

We claim that $A_N f \to E(f)$ as $N \to \infty$.

Let's first try to show this for $f \in C(\mathbb{T})$. Note that this property is linear and behaves well under L^∞ approximation, so consider $f_n \to f$ in L^∞ and $A_N f_n \to E(f_n)$. Let $\varepsilon > 0$. There exists n such that $A_N(f - f_n) < \varepsilon$ and $|E(f_n) - E(f)| < \varepsilon$. This gives us convergence for f also.

Thus we need only show the result for trig polynomials $f_k(x) = e^{2\pi i k x}$.

We calculate $A_N f$ directly. If $k = 0$, the result is trivial, so assume $k \neq 0$.

$$A_N f = \frac{1}{N} \sum_{n=0}^{N} \exp(2\pi i k \alpha)^n = \frac{1}{N} \frac{1 - \exp(2\pi i N k \alpha)}{1 - \exp(2\pi i k \alpha)} \to 0$$

as $N \to \infty$. Note that $E(f_k) = 0$ for $k \neq 0$.

To apply the result to $\chi_{[a,b]}$ we need to approximate by continuous functions. Take f_n, g_n continuous such that $0 \le g_n \le \chi_{[a,b]} \le f_n \le 1$ with $f_n, g_n \to \chi_{[a,b]}$ Lebesgue almost everywhere.

Then

$$A_N g_n \le A_n \chi_{[a,b]} \le A_N f_n$$
$$E(g_n) \le E(\chi_{[a,b]} \le E(f_n)$$

Sending $N \to \infty$ gives us $E(g_n) \le \liminf_{N\to\infty} A_N\chi \le \limsup_{N\to\infty} A_N \le E(f_n)$. The dominated convergence theorem with $n \to \infty$ gives us the desired result.

- Part 3 Method 1

  Let μ be an α-invariant Borel probability measure on $\mathbb{T}$. It's enough to show that $\int f\, d\mu = \int f\, dm$ for all $f \in C(\mathbb{T})$.

  We calculate

$$\begin{aligned}\int f(x)\, d\mu(x) - \int f(z)\, dm(z) &= \iint (f(x) - f(z))\, dm(z)\, d\mu(x)\\ &= \iint (f(x) - f(x+z))\, dm(z)\, d\mu(x)\\ &= \iint (f(x) - f(x+z))\, d\mu(x)\, dm(z)\end{aligned}$$

  where we apply translation invariance of m and Fubini.

  Thus it's enough to show the integrand $\int f(x) - f(x+z)\, d\mu(x)$ is zero for almost every $z \in \mathbb{T}$. (We'll show this for every $z \in \mathbb{T}$).

  By density, there exists $n_j\alpha \to z$ as $j \to \infty$. Then $f(x + n_j\alpha) \to f(x+z)$ uniformly since f is continuous and $\mathbb{T}$ is compact (and hence f is uniformly continuous).

  Thus

$$\int f(x) - f(x+z)\, d\mu(x) = \int f(x + n_j\alpha) - f(x+z)\, d\mu(x) \to 0$$

  as $j \to \infty$ by uniform convergence.

  Thus $\int f\, d\mu = \int f\, dm$ for all continuous f, so $m = \mu$.

- Part 3 Method 2 Suppose α is irrational. Then if f is a trig polynomial, the same calculation as before shows that

$$A_N f(x) := \frac{1}{N}\sum_{n=0}^{N-1} f(x + n\alpha) \to \int_{\mathbb{T}} f\, dm$$

  as $N \to \infty$ for **any** fixed $x \in \mathbb{T}$ (this is the pointwise ergodic theorem but at **every** point!). Let μ be any \$R$_\alpha$\$-invariant measure. Then as trig polynomials are bounded, the dominated convergence theorem implies

$$\int A_N f\, d\mu \to \int \left(\int f\, dm\right) d\mu = \int f\, dm$$

but since μ is R_α invariant, the left hand side is equal to $\int f \, d\mu$ for all N. Thus $\int f \, d\mu = \int f \, dm$ for all trig polynomials f, and hence for all $f \in C(\mathbb{T})$ by density. Thus the Riesz representation theorem gives $\mu = m$.

Problem 5

Let $\{f_n\}$ be a sequence of continuous real-valued functions on $[0, 1]$ and suppose $f_n(x)$ converges to another real valued function $f(x)$ at **every** $x \in [0, 1]$.

1. Prove that for every $\varepsilon > 0$ there is a dense subset $D_\varepsilon \subseteq [0, 1]$ such that if $x \in D_\varepsilon$, then there is an open interval $I \ni x$ and a positive integer N_x such that $\forall n > N_x$, $\sup_I |f_n - f| \leq \varepsilon$.
2. Prove that f cannot be the characteristic function $\chi_{\mathbb{Q} \cap [0,1]}$.

Proof

- Part 1

  Let $\varepsilon > 0$. For $N \in \mathbb{N}$, consider

  $$C_N = \{x \in [0, 1] \mid \forall n > N. \ |f_n(x) - f(x)| \leq \varepsilon\}.$$

  Note that $C_N \subseteq C_M$ for $N \leq M$, that $\bigcup_{N \in \mathbb{N}} C_N = [0, 1]$ by pointwise convergence, and that C_N is closed for each N, since

  $$C_N = \bigcap_{n > N} |f_n - f|^{-1}([0, \varepsilon])$$

  which is an intersection of the preimages of closed sets under continuous functions.

  Note that since $[0, 1]$ is complete, at least one of the C_N is not nowhere dense by the Baire Category Theorem. In particular, C_N° is not empty. Also C_N° is a union of intervals. Let's call these intervals $I_N^1, \ldots I_N^{n_N}$.

  For any $x \in I_N^i$ for any i and any $n > N$, we have $\sup_{y \in I_N^i} |f_n(y) - f(y)| \leq \varepsilon$.

  Consider a dense subset of each interval to form the first part of D_ε.

  Now we apply again the Baire Category Theorem to $[0, 1] \setminus \left(\bigcup I_N^i\right)$ to get more intervals and another value of N.

  If our intervals are not dense in $[0, 1]$, then there is some interval $(a, b) \supseteq [a + \eta, b - \eta]$ on which we may apply the Baire Category theorem again.

- Part 2

  Let $\varepsilon = 1/3$. Then on no interval is $\sup|f_n(y) - f(y)| \leq \varepsilon$ for any continuous function f_n.

Problem 6

Let $f \in L^2(\mathbb{R})$ and assume the fourier transform satisfies $\left|\hat{f}(\xi)\right| > 0$ for Lebesgue a.e. $\xi \in \mathbb{R}$.

Prove the set of finite linear combinations of the translates $f_y(x) = f(x - y)$ is norm dense in $L^2(\mathbb{R})$

Proof

Let $M = \overline{\operatorname{span}\{f_a\}_{a\in\mathbb{R}}}$ where the closure is with respect to L^2. Suppose for contradiction that $M \neq L^2$.

Then there is a nonzero $g \in M^\perp$, so in particular

$$\int_{\mathbb{R}} f(x-a)g(x)\,dx = 0$$

for all $a \in \mathbb{R}$. By Plancherel,

$$\begin{aligned}\int \hat{f}_a(\xi)\hat{g}(\xi)\,d\xi &= \int_{\mathbb{R}} e^{-2\pi i a\xi}\hat{f}\hat{g}\,d\xi \\ &= \widehat{\hat{f}\hat{g}}(a) = 0\end{aligned}$$

for all a, where $\hat{f}\hat{g} \in L^1$ and so that Fourier transform makes sense. Since for L^1 functions we have Fourier inversion, $\hat{f}\hat{g} = 0$ everywhere.

Thus, since $\left|\hat{f}\right| > 0$, we must have $\hat{g} = 0$, and so $g = 0$.

Problem 7

Let $f(z)$ be an analytic function on the entire complex plane $\mathbb{C}$ such that the function $U(z) = \log|f(z)|$ is Lebesgue area integrable. Prove that f is constant.

Proof

Suppose f is not constant. Then f isn't bounded, so there exists $z_0 \in \mathbb{C}$ such that $\log|f(z_0)| > 1$. Recall that $\log|f|$ is subharmonic, so by the mean value property,

$$\int_{\mathbb{R}^2} \log|f(z)|\, d\lambda = \int_0^\infty r \int_0^{2\pi} \log\left|f(z_0 + re^{i\theta})\right| d\theta\, dr$$
$$\geq \int_0^\infty 2\pi r\, dr = \infty$$

This is a contradiction ↯

Problem 8

Let $\mathscr{D}$ be the space of analytic functions $f(z)$ on the unit disc $\mathbb{D}$ such that $f(0) = 0$ and $\int_{\mathbb{D}} |f'(z)|^2\, dx\, dy < \infty$.

1. Prove that $\mathscr{D}$ is complete in the norm $\|f\| = \|f'\|_{L^2}$.

2. Give a necessary and sufficient condition on the coefficients a_n for the function $f(z) = \sum_{n \geq 1} a_n z^n$ to belong to $\mathscr{D}$.

Proof

- Part 1 Let f_n be Cauchy in $\mathscr{D}$. Then f_n' is Cauchy in $L^2(\mathbb{D})$, and so there is a convergent subsequence $f_n' \to g$ in L^2.

  Our goal is that g is holomorphic. We know that a locally uniformly bounded family of holomorphic functions is normal. This may help us.

  There is a standard trick. We write

  $$|f'(z)| = \left| \int_{B(z,(1-r)/2)} f'(w)\, dA(w) \right|$$
  $$\leq \int_{B(z,(1-r)/2)} |f'(w)|\, dA(w)$$
  $$\lesssim_r \|f'\|_{L^2(B(z,(1-r)/2))} \leq \|f\|$$

  This means that $\|f'\|_{L^\infty(B(0,r))} \leq \|f\|$.

  Since the former space is complete, we know that in fact the f_n' converge locally uniformly. But $f_n' \to g$ in L^2 (hence pointwise ae along a subsequence), so g is the uniform on compact sets limit of holomorphic functions, and hence holomorphic!

  Now we define G as the unique antiderivative of g with $G(0) = 0$ and we're done.

  A quick overview of the process:

1. A bound on the L^1 norm of the function bounds the value of the function itself by the mean value property.
2. On small sets, this means a bound on L^2 can bound L^∞
3. Convergence in L^∞ is uniform convergence.
4. Uniform convergence on compact sets of holomorphic functions means there's a holomorphic limit.

- Part 2

  We write

$$\begin{aligned}
\int_{\mathbb{D}} |f'(z)|^2 \, dA &= \int_{\mathbb{D}} \left(\sum_{n\geq 1} a_n z^n \right) \left(\sum_{m \geq 1} \overline{a_m} \overline{z}^m \right) \, dA \\
&= \int_0^1 \int_0^{2\pi} \text{ terms times } z^{\text{stuff}} \; + \; \text{terms times } \overline{z}^{\text{stuff}} \; d\theta \, dr \\
&\quad + \int_0^1 \int_0^{2\pi} \sum n^2 |a_n|^2 r^{2n-1} \, d\theta \, dr
\end{aligned}$$

  and we can swap everything by appropriate uses of monotone convergence and uniform convergence on compact sets. More carefully:

$$\begin{aligned}
\int_{\mathbb{D}} |f'|^2 &= \int_0^1 \int_0^{2\pi} \sum_{n,k\geq 1} nk a_n \overline{a_k} r^{n+k-2} e^{i(n-k)\theta} r \, d\theta \, dr \\
&= \int_0^1 \sum nj a_n \overline{a_k} r^{n+k-1} \int_0^{2\pi} e^{i(n-k)\theta} \text{ because the series converges unif on cpt sets} \\
&= \int_0^1 \sum_{n\geq 1} n^2 |a_n|^2 r^{2n-1} \, dr \\
&= \sum_{n \geq 1} n^2 |a_n|^2 \int_0^1 r^{2n-1} \, dr \text{ by monotone convergence} \\
&= \frac{1}{2} \sum_{n\geq 1} n |a_n|^2
\end{aligned}$$

Problem 9

Consider the meromorphic function $g(z) = -\pi z \cot(\pi z)$ on the entire complex plane $\mathbb{C}$.

1. Find all the poles of g and determine the residue of g at each pole.

2. In the Taylor series representation $\sum_{k=0}^{\infty} a_k z^k$ of $g(z)$ about 0, show that for each $k \geq 1$,

$$a_{2k} = \sum_{n \geq 1} \frac{2}{n^{2k}}$$

Proof

- Part 1

  First, write

  $$g(z) = \frac{-\pi z \cos(\pi z)}{\sin(\pi z)}$$

  and so g has simple poles at each integer.

  To measure the residues, we calculate

  $$\lim_{z \to n} \frac{-\pi z (z-n) \cos(\pi z)}{\sin(\pi z)} = -n(-1)^n = (-1)^{n+1} n$$

- Part 2

  Now we consider the other representation:

  $$\begin{aligned} \pi \cot(\pi z) &= \sum_{k=-\infty}^{\infty} \frac{1}{z-k} \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}. \end{aligned}$$

  Thus

  $$g(z) = -1 - \sum_{k=1}^{\infty} \frac{2z^2}{z^2 - k^2}$$

  We can write this as a power series in z^2.

  $$\begin{aligned} g(z) &= f(z^2) \\ f(z) &= -1 - \sum_{k=1}^{\infty} \frac{2z}{z - k^2} \end{aligned}$$

Note that f is holomorphic except where $z = k^2$ since the series converges locally uniformly. Thus the coefficient to z^2 for g is the coefficient to z in f.

All we need to do is differentiate f now. The product rule requires we take exactly one derivative of z and get

$$f^{(n)}(0) = -2\left(\sum_{k=1}^{\infty} \frac{1}{z-k^2}\right)^{(n-1)}$$

which we can just evaluate term-by-term (since the series converges uniformly on compact sets).

Problem 10

For $-1 < \beta < 1$, evaluate

$$\int_0^{\infty} \frac{x^\beta}{1+x^2}\, dx$$

Proof

We take a branch cut of log with the negative imaginary axis removed (so that the imaginary part of log goes from $-\pi/2$ to $3\pi/2$). We then take the contour $1/R$ to R along $\mathbb{R}^+$, then backwards along a semicircle to $-R$, then straight to $-1/R$, then another semi-circle to $1/R$.

Note that

$$\frac{z^\beta}{1+z^2} = \frac{\exp(\beta \log z)}{1+z^2}$$

goes to zero on the small semicircle and decays faster than $1/R$ on the large semi-circle. Thus only the real axis contributions matter.

Now for the residues, note that $1+z^2 = (-i+z)(i+z)$ which has exactly one pole in the upper half-plane: i. The residue then is

$$\frac{\exp(\beta \log i)}{2i} = \frac{\exp(\beta i\pi/2)}{2i}$$

Therefore

$$\int_{-\infty}^{0} \frac{\exp(\beta \log z)}{z^2+1}\, dz + \int_0^\infty \frac{x^\beta}{x^2+1}\, dx = 2\pi i \left(\frac{\exp(\beta i\pi/2)}{2i} \right)$$
$$\left(1 + e^{\pi \beta i}\right) \int_0^\infty \frac{x^\beta}{x^2+1}\, dx = \pi \exp(\beta i \pi/2)$$
$$\int_0^\infty \frac{x^\beta}{x^2+1}\, dx = \frac{\pi \exp(\beta i\pi/2)}{1+\exp(\beta i \pi)}$$

Cancel and view it as reciprocal of cosine. We get

$$\frac{\pi}{2\cos(\beta\pi/2)}$$

Problem 11

An analytic Jordan curve is a set of the form $\Gamma = f(\{|z| = 1\})$ where f is analytic and injective on an annulus $\{r < |z| < 1/r\}$ (where $0 < r < 1$).

Let $\mathbb{C}^*$ be the Riemann sphere, $N < \infty$, and $\Omega \subseteq \mathbb{C}^*$ be a domain for which $\partial\Omega$ has N connected components, none of which are single points.

Prove there is a conformal map from Ω onto a domain bounded by N pairwise disjoint analytic Jordan curves.

Proof

Consider $\Omega^c = \bigcup E_i$ for $i = 1, \dots, N$ where E_N is the unbounded component.

For each E_i for $i < N$, take a conformal mapping of $\mathbb{C}^* \setminus E_i$ that takes this region to the complement of a disc. This is possible by the Riemann mapping theorem.

Now do this for every other region (which is now modified). We've turned the boundary of region E_1 into a Jordan curve, and all the others will become Jordan as well.

The "single point" assumption is precisely what's needed to make the Riemann mapping argument work.

Problem 12

If $\alpha \in \mathbb{C}$ satisfies $0 < |\alpha| < 1$ and $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, show that the equation

$$e^z(z-1)^n = \alpha$$

has exactly n simple roots in the half-plane $\{z \mid \operatorname{Re} z > 0\}$.

Proof

First, note that $e^z(z-1)^n = 0$ has n roots in the right half-plane: 1 with multiplicity n. Let's show that $f(z) = e^z(z-1)^n$ has as many roots as $g(z) = e^z(z-1)^n - \alpha$.

Let's apply the symmetrized Rouche theorem in a semi-circle whose diameter lies on the imaginary axis. We need to verify that

$$\begin{aligned} |f-g| &< |f| + |g| \\ |\alpha| &< |e^z(z-1)^n| + |e^z(z-1)^n - \alpha| \end{aligned}$$

along the boundary.

Note that $|\alpha| < 1$. Furthermore, writing $z = x + iy$, we have

$$|e^z(z-1)^n| = |e^x(z-1)^n| \qquad \geq |z-1|^n$$

which goes to ∞ on the semi-circle as the radius goes to ∞.

Next, along the imaginary axis we have

$$|e^z(z-1)^n| = |(z-1)^n| = |-1-iy|^n \geq 1 > |\alpha|$$

and so Rouché's theorem holds.

Note: we could have used the regular Rouche theorem here, but the symmetrized one is stronger and lets us be unsure of the functions when we start.

Next we verify that the roots are all simple. Suppose $e^z(z-1)^n = \alpha$. Then

$$\begin{aligned} g'(z) &= e^z(z-1)^n + ne^z(z-1)^{n-1} \\ &= \alpha + ne^z(z-1)^{n-1} \end{aligned}$$

If this is zero, then $z - 1 = -n$. But $\operatorname{Re} z > 0$ and $n \geq 1$, so this is impossible.

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Problem 1

Let $f \in C^2(\mathbb{R})$ be a real valued function that is uniformly bounded on $\mathbb{R}$. Prove that there exists a point $c \in \mathbb{R}$ such that $f''(c) = 0$.

Proof

Suppose not. Without loss of generality, assume $f'' > 0$ always. Let x be a point such that $f'(x) > 0$ or y be a point such that $f'(y) < 0$.

Note that for all $z > x$, $f'(z) > f'(x)$. Thus $f(z) \geq f(x) + f'(x)(z-x)$ and thus goes to ∞. Similarly we can analyze $z < y$.

Problem 2

Let μ be a Borel probability measure on $[0, 1]$ that has no atoms. Let $\mu_1, \ldots$ be Borel probability measures on $[0, 1]$. Assume that $\mu_n \to \mu$ in the weak-star topology.

Denote $F(t) = \mu([0, t])$ and $F_n(t) = \mu_n([0, t])$. Prove that $F_n \to F$ uniformly.

Proof

Let $\varepsilon > 0$. Since μ has no atoms, we can take finitely many points $x_1 < \cdots < x_M$ such that $\mu([x_i, x_{i+1}]) < \varepsilon$.

Note that $F_n \to F$ pointwise by the Portmanteau theorem (or a simple argument involving continuous functions approximating $[0, t]$). Then for $x_{i-1} < s < x_i$, we have

$$\begin{aligned} |(F_n - F)(s) - (F_n - F)(x_i)| &\leq |\mu_n([s, x_i]) - \mu([s, x_i])| \\ &\leq |\mu_n([s, x_i]) - \varepsilon| \\ &\leq \mu_n([x_{i-1}, x_i]) + \varepsilon \end{aligned}$$

whose limsup is at most 2ε at $n \to \infty$. Thus take N_i such that for all $n > N_i$, the above quantity is at most 3ε

Thus when we take N_i even larger so that $|(F_n - F)(x_i)| < \varepsilon$, we can take the maximum of each of these integers and get some N such that for all $s \in [0, 1]$, when $n > N$, we have $|(F_n - F)(s)| \leq 4\varepsilon$. This is uniform convergence.

Problem 3

1. Let f be a positive continuous function on $\mathbb{R}$ such that $\lim_{|t| \to \infty} f(t) = 0$, i.e., f vanishes at infinity.

   Show that the set $\{hf \mid h \in L^1(\mathbb{R}), \|h\|_{L^1} \leq K\}$ is a closed nowhere dense set in L^1 for any $K \geq 1$.

2. Let $\{f_n\}$ be a sequence of positive continuous functions on $\mathbb{R}$ that vanish at infinity. Show that there exists $g \in L^1$ such that $g/f_n \notin L^1$ for any n.

Proof, Part 1

First, closedness. Suppose $h_n f \to g$ in L^1.

Then $\int |g/f| \leq \liminf \int |h_n| \leq K$ by Fatou's lemma, so the set is closed.

For nowhere density: it suffices to show there is $g \in L^1$ with small norm, but g/f has arbitrarily large norm.

First, note that eventually, $1/f$ is at least magnitude 1^2. On an interval of width 1 past this point, have g take the value 1.

Next, $1/f$ is eventually at least 2^2 on an interval of width 1 past the first interval. Have g take the value $1/4$ here.

Repeat as often as needed, and scale g appropriately so its L^1 norm is small, but g/f has large L^1 norm.

Proof, Part 2

Next, partition the line into $E_1 \sqcup \cdots \sqcup E_n \sqcup \ldots$ such that each set is infinite in measure and consists of intervals of length 1 (for example, E_1 is every second interval, and E_n is every second interval of the remaining ones after the first $n-1$ sets are removed).

Do the construction above restricted to each E_n and make sure the norm of g is still small, but now make sure g/f_n on E_n has infinite norm.

Problem 4

Let V be the subspace of $L^\infty([0,1], \mu)$ (where μ is Lebesgue measure) defined by

$$V = \left\{ f \in L^\infty \mid \lim_{n\to\infty} n \int_{[0,1/n]} f \text{ exists} \right\}$$

1. Prove that there exists $\phi \in (L^\infty)^*$ such that $\phi(f) = \lim n \int_{[0,1/n]} f$ for every $f \in V$.

2. Show that given any $\phi \in (L^\infty)^*$ satisfying the condition above, there exists no $g \in L^1$ such that $\phi(f) = \int fg$ for all $f \in L^\infty$.

Proof, Part 1

Define $\phi : V \to \mathbb{R}$ via $\phi(f) = \lim n \int_{[0,1/n]} f$. We need to verify that this is continuous. Note that

$$|\phi(f)| \le \limsup_n \|f\|_{L^\infty} \cdot n/n = \|f\|_{L^\infty}$$

and so this functional can be extended by Hahn-Banach.

Proof, Part 2

Let ϕ be any such functional. Suppose $\phi(f) = \int fg$ for all $f \in L^\infty$.

Consider $f_\alpha = \alpha \chi_{[1/n,1]}$. Note that $\phi(f_\alpha) = 0$ for all α, so $g = 0$ on $[1/n, 1]$, which is true for every n, so $g = 0$, a contradiction.

Problem 5

1. Prove that L^p is a separable Banach space for $1 \le p < \infty$ but L^∞ is not separable.

2. Prove that there exists no linear bounded surjective map $T : L^p \to L^1$ if $p > 1$.

Proof, Part 1

These are easily shown to be Banach spaces. Recall that to show a space is complete, it's enough to show that absolutely convergent series converge. So, take $\sum \|f_n\| < \infty$. Then

$$\left(\int \left(\sum |f_n| \right)^p \right)^{1/p} \le \sum \|f_n\| < \infty$$

so $f = \sum f_n \in L^p$. Then the dominated convergence theorem finishes the job.

For separability in L^p, consider that Schwartz functions are dense in L^p, that the space of polynomials restricted to $[-n, n]$ is dense in Schwartz functions, and that polynomials with rational coefficients on $[-n, n]$ are dense in all polynomials in the same interval.

For failure of separability in L^∞, consider the family of functions $\chi_{[0,r]}$. These are all 1 apart in norm, and there are uncountably many of them. Separability is clearly impossible.

Proof, Part 2

Suppose $T : L^p \to L^1$ is bounded, surjective, and linear. The adjoint, $T^* : L^\infty \to L^{p'}$ is thus bounded, injective, and linear. But then T^* is an isomorphism from L^∞ to a subspace of $L^{p'}$ and thus a subspace of $L^{p'}$ is not separable, which is impossible.

Problem 6

Let H be a Hilbert space and $\{\xi_n\}$ a sequence of vectors with norm 1.

1. Show that if $\xi_n \rightharpoonup \xi$ with $\|\xi\| = 1$, then $\xi_n \to \xi$ strongly.

2. Show that if $\lim_{n,m\to\infty} \|\xi_n + \xi_m\| = 2$, then there exists ξ such that $\xi_n \to \xi$.

Proof, Part 1

Write

$$\begin{aligned}\|\xi_n - \xi\|^2 &= \langle \xi_n - \xi, \xi_n - \xi\rangle \\ &= \langle \xi_n, \xi_n\rangle - 2\operatorname{Re}\langle \xi, \xi_n\rangle + \langle \xi, \xi\rangle \\ &= 2 - 2\operatorname{Re}\langle \xi, \xi_n\rangle \to 0\end{aligned}$$

since $\langle \xi, \xi_n\rangle \to 1$.

Proof, Part 2

By Banach Alaoglu, there's a weakly convergent subsequence $\xi_{n,k} \rightharpoonup \xi$. Note that

$$\begin{aligned}\left\|\xi_{n_k} + \xi_{n_j}\right\|^2 &= \left\langle \xi_{n_k} + \xi_{n_j}, \xi_{n_k} + \xi_{n_j}\right\rangle \\ &= 2 + 2\operatorname{Re}\left\langle \xi_{n_j}, \xi_{n_k}\right\rangle \to 4\end{aligned}$$

If we set $j \to \infty$ first, we get $2 + 2\operatorname{Re}\left\langle \xi, \xi_{n_k}\right\rangle \to 4$. Thus we know actually that $\|\xi\| = 1$. Thus $\xi_{n_k} \to \xi$ strongly.

Now we note that $\lim_{k,m\to\infty}\left\|\xi_{n_k} + \xi_m\right\| = 2$, so $\lim_{m\to\infty}\|\xi + \xi_m\| = 2$.

Now suppose $\xi_{m_j} \to x$ is any other similarly produced strongly convergent sequence. Then $\|\xi + x\| = 2$, so $\xi = x$ by the reverse triangle inequality.

Thus every sequence has a subsequence which converges strongly (and to the same vector!)

Much Better Proof, Part 2

We apply the parallelogram law:

$$\|\xi_n + \xi_m\|^2 = 2\|\xi_n\|^2 + 2\|\xi_m\|^2 - \|\xi_n - \xi_m\|^2$$

The left hand size converges to 4 and the right hand side converges to $4 - \lim\|\xi_n - \xi_m\|^2$, so we have a Cauchy sequence.

Problem 7

Let $f : \mathbb{C} \to \mathbb{C}$ be entire and non-constant, and let us set

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} \log_+\left|f(re^{i\phi})\right| d\phi$$

Here $\log_+ = \max(\log, 0)$. Show that $T(r) \to \infty$ as $r \to \infty$.

Proof

Recall Jensen's formula:

$$\fint_0^{2\pi} \log\left|f(re^{i\phi})\right| d\phi = \sum \log \frac{r}{|a_i|} + \log|f(0)|$$

where a_i enumerates zeroes with $|a_i| \leq r$.

Note that as $r \to \infty$, the sum must also diverge if there are any zeroes. Thus if f has any zeroes, we must have that $\fint \log_+\left|f(re^{i\phi})\right| d\phi$ diverges as well, a contradiction.

Thus f has no zeros.

If f has no zeroes, we consider $f - \varepsilon$ which does have zeroes. Note that $\log|f - \varepsilon| \leq \log(|f| + \varepsilon) \leq \log_+(|f| + \varepsilon)$.

In particular,

$$\begin{aligned}\log_+(|f| + \varepsilon) &= \begin{cases} 0 & |f| \leq 1 - \varepsilon \\ \log(|f| + \varepsilon) = \log|f| + \log(1 + \varepsilon/|f|) \leq \log_+|f| + 1 + \varepsilon/(1 - \varepsilon) & |f| + \varepsilon > 1 \end{cases} \\ &\leq \max\left(\log_+|f| + 1 + \varepsilon/(1 - \varepsilon), 0\right)\end{aligned}$$

and so we can bound

$$\begin{aligned}\fint_0^{2\pi} \log_+|f - \varepsilon|\, d\theta &\leq \fint \max\left(\log_+|f| + 1 + \varepsilon/(1 - \varepsilon), 0\right)\, d\theta \\ &\leq \fint \log_+|f| + 1 + \varepsilon/(1 - \varepsilon)\end{aligned}$$

Since the LHS goes to infinity, so does the right hand side.

Problem 8

Show that

$$\sin z - z \cos z = \frac{z^3}{3} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

where λ_n is a sequence in $\mathbb{C}$ with $\lambda_n \neq 0$ for all n such that $\sum \frac{1}{|\lambda_n|^2} < \infty$.

Proof

Recall that if λ is the order and h the genus, then $h \leq \lambda \leq h + 1$. The order is $\limsup_{r \to \infty} \log\log M(r)/\log r$. We can bound $|\sin z - z\cos z| \leq (1 + |z|)\left|\exp z\right|$, and so the order is at most 1 and hence the genus is at most 1 too.

In particular, the genus is at most 1, so we have the following product expansion

$$\sin z - z\cos z = Cz^k e^{g(z)} \prod_{i=1}^{\infty}(1 - \frac{z}{\lambda_i})\exp(z/\lambda_i)$$

where the zeroes must satisfy $\sum 1/|\lambda_i|^2 < \infty$.

Note that $\sin z - z\cos z$ has a zero of order 3 at the origin, so $k = 3$ and the constant $C = \frac{1}{3}$ by computation.

Finally, we observe that $\sin z - z\cos z$ is odd, so the zeroes pair off and we combine to get

$$\sin z - z\cos z = \frac{z^3}{3}e^{g(z)} \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right)$$

where the exponentials cancel out.

Finally, since everything is odd, we must have $e^{g(z)}$ is even. Because this is genus 1, we have $g(z) = az$ for some a, and this is never even unless $a = 0$, so we have the desired product expansion.

Problem 9

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and let $A(\mathbb{D})$ be the space of functions holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Let

$$U = \{f \in A(\mathbb{D}) \mid |f(z)| = 1 \text{ on } \partial\mathbb{D}\}$$

Show that $f \in U$ iff f is a finite Blashcke product.

Proof

Let $B(z)$ be a Blaschke product with the same zeroes. Then consider f/B. This is holomorphic in $\mathbb{D}$ and continuous up to the boundary. Furthermore, it lacks zeroes.

If f/B is not a constant, then for some $z \in \mathbb{D}$, $|f/B| < 1$. But now consider B/f which lacks zeroes, is holomorphic, and has a maximum on the interior of $\mathbb{D}$. This is a contradiction, so $f = \lambda B$ for some constant λ. Clearly $|\lambda| = 1$ by considering the modulus on the boundary circle.

Problem 10

For $a > 0, b > 0$ evaluate the integral

$$\int_0^\infty \frac{\log x}{(x+a)^2 + b^2}\,dx$$

Proof

First, take a branch cut of log along the positive reals.

Consider the keyhole integral from $\varepsilon + i\varepsilon$ to $R + i\varepsilon$, then around most of a circle to $R - i\varepsilon$, back along a line to $\varepsilon - i\varepsilon$, then around a circle to $\varepsilon + i\varepsilon$. The integral on the small circle goes to zero (we bound by by $\log x \cdot x \to 0$). The integral on the large circle goes to zero (since $\log x/x \to 0$ as $x \to \infty$). Thus, we're left with

$$\begin{aligned}(1+2\pi i)\int_0^\infty &= \int_\gamma = 2\pi i \sum \operatorname{Res} \\ &= 2\pi i\left(\frac{\log(-a-bi)}{-2bi} + \frac{\log(-a+bi)}{2bi}\right) \\ &= 2\pi\left(\frac{\log(-a-bi)}{-2b} + \frac{\log(-a+bi)}{2b}\right)\end{aligned}$$

which is real.

Problem 11

Let $u \in C^\infty(\mathbb{R})$ be smooth 2π-periodic. Show that there exists a bounded holomorphic function f_+ in the upper half plane and a bounded holomorphic function f_- in the lower half-plane such that

$$u(x) = \lim_{\varepsilon \to 0}\left(f_+(x+i\varepsilon) - f_-(x-i\varepsilon)\right)$$

Proof

First, we solve the Dirichlet problem in the upper half-plane with $u/2$. We get a harmonic function g_+ in the upper half plane with some harmonic conjugate h_+. Write $f_+ = g_+ + ih_+$.

Then write $f_- = \overline{-f_+(\bar{z})}$. This way, $\operatorname{Re}(f_-(x)) = -\operatorname{Re}(f_+(x))$ for $x \in \mathbb{R}$, but the imaginary part has the same sign.

Observe too that f_- is holomorphic in the lower half plane.

Thus $f_+(x+i\varepsilon) - f_-(x-i\varepsilon)$ goes to $u(x)/2 + u(x)/2 = u(x)$.

Problem 12

Let H be the vector space of entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\int_{\mathbb{C}} |f(z)|^2 \, d\mu(z) < \infty$$

where $d\mu(z) = e^{-|z|^2} \, d\lambda(z)$ where λ is the Lebesgue measure.

1. Show that H is a closed subspace of $L^2(\mathbb{C}, d\mu)$.

2. Show that for all $f \in H$, we have

$$f(z) = \frac{1}{\pi} \int_{\mathbb{C}} f(w) e^{z\bar{w}} \, d\mu(w)$$

   Hint: Show that the normalized monomials

$$e_n(z) = \frac{1}{(\pi n!)^{1/2}} z^n \quad n = 0, 1, \ldots$$

   form an orthonormal basis of H.

Proof, Part 1

Let $f_n \to f$ in $L^2(\mathbb{C}, d\mu)$ where $f_n \in H$. It suffices to prove that $f_n \to f \in L^2_{\text{loc}}(d\lambda)$, since then it will converge locally in L^1 and thus locally uniformly by a standard application of the mean value theorem:

$$|f_n(z) - f_m(z)| = \frac{1}{2\pi r} \int_{B(z,r)} |f_n - f_m| \, d\lambda \to 0$$

so the sequence $\{f_n\}$ is Cauchy (and in fact this is locally uniform). Since the locally uniform limit of holomorphic functions is holomorphic by an application of Morera's theorem, f is holomorphic.

Now to show locally L^2 convergence.

Pick $R > 0$. Then $0 \leftarrow \int_{\mathbb{C}} |f - f_n|^2 e^{-|z|^2} \lambda \geq e^{-R^2} \int_{B(0,R)} |f - f_n|^2$, so $f_n \to f$ in $L^2_{\text{loc}}(d\lambda)$.

Thus f is holomorphic, since it's the locally uniform limit of holomorphic functions.

Proof, Part 2

It's clear that the normalized monomials are orthogonal since their products are monomials of positive degree which on every circle have integral zero. To show they're actually normalized, we calculate

$$\begin{aligned}\int_{\mathbb{C}} \frac{1}{\pi n!} z^{2n} e^{-|z|^2}\, d\lambda(z) &= \frac{1}{\pi n!} \int_0^\infty \int_0^{2\pi} r^{2n} e^{-r^2} r\, d\theta\, dr \\ &= \frac{2}{n!} \int_0^\infty r^{2n+1} e^{-r^2}\, dr \\ &= \frac{1}{n!} \int_0^\infty s^n e^{-s}\, ds = 1\end{aligned}$$

by a change of variables and viewing the last integral as the gamma function.

Thus these normalized monomials are in fact orthonormal. Furthermore, they are a complete orthonormal system.

Finally, we verify the desired equality for all normalized monomials. We compare $\frac{z^n}{(\pi n!)^{1/2}}$ to

$$\begin{aligned}\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{(\pi n!)^{1/2}} w^n e^{z\overline{w}}\, d\mu(w) &= \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{(\pi n!)^{1/2}} w^n e^{z\overline{w}} e^{-|w|^2}\, d\lambda(w) \\ &= \frac{1}{\sqrt{\pi^3 n!}} \int_{\mathbb{C}} w^n e^{(z-w)\overline{w}}\, d\lambda(w) \\ &= \frac{1}{\sqrt{\pi^3 n!}} \int_{\mathbb{C}} (\zeta + z)^n e^{\zeta \overline{(z-\zeta)}}\, d\lambda(\zeta)\end{aligned}$$

Now that we have the basis, we can calculate with a standard trick what the integrand should be.

First, note that the operator $f \mapsto f(w)$ is continuous as an L^2 operator:

$$\begin{aligned}|f(w)| &\le \left| \int_{B(w,(r-|w|)/2)} f(z)\, d\lambda \right| \\ &\lesssim \|f\|_{L^2(B(0,r),d\lambda)} \\ &\lesssim \|f\|_{L^2(B(0,r),d\mu)} \\ &\lesssim \|f\|_{L^2(\mathbb{C},d\mu)}\end{aligned}$$

by the mean value property.

Next, since $f \mapsto f(w)$ is a continuous L^2 operator, it is represented by a function $g_w(z)$. We can discover g_w as follows:

$$\begin{aligned}
g_w(z) = \langle g_w, g_z \rangle &= \sum_n \langle g_w, e_n \rangle \overline{\langle g_z, e_n \rangle} \\
&= \sum_n \overline{e_n(w)} e_n(z) \\
&= \frac{1}{\pi} \sum_n \frac{1}{n!} \overline{w}^n z^n \\
&= \frac{1}{\pi} e^{z\overline{w}}
\end{aligned}$$

as desired.