GEOMETRY TOPICS TO KNOW OR LOOK UP

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Differential Topology

(1) Submersions, immersions, embeddings, and local submersion/immersion theorems about coordinates

Locally, immersions (df injective) look like the canonical immersion $a \mapsto (a, 0)$. Proof: for a function f, let it be g in local coordinates, so that $dg = \binom{I}{0}$ up to change of basis. Then $G(x, z) = g(x) + (0, z)$ has identity as its differential. Apply inverse function theorem.

Similarly, local submersions use inverse function theorem with the helper function $G(a) = (g(a), a_{k+1}, \ldots, a_n)$.

Embeddings are topological embeddings (homeo onto image) which are also injective immersions. A stricter condition which is used more often is proper ($f^{-1}(K)$ is compact) injective immersion.

(2) Sard's Theorem

Almost every value is regular. The benefit being that the preimage of a regular value of $f : X \to Y$ is a manifold of dimension $\dim X - \dim Y$.

(3) Covering map and how to show. A covering map is a map where the stack of records theorem holds: i.e., for $f : X \to Y$, for any $y \in Y$, there exists $V \ni y$ neighborhood such that for each $x \in f^{-1}(y)$, there exists $U_x \ni x$ diffeomorphic to V via f. (Alternatively, homeomorphic).

A fiber bundle with discrete fiber is exactly a covering map, so if f is a proper submersion on a connected space between equidimensional manifolds, it is (by Ehresmann or an easy proof) a covering map.

(4) Ehresmann's fibration theorem: a proper submersion over a connected space is a fiber bundle. The proof sucks.

(5) Intersection number: $f \pitchfork Z$ implies $I(f, Z) = \sum_{x \in f^{-1}(Z)}$ orientation # at x.

$$\text{orientation \# at } x = \begin{cases} +1 & \text{if } df_x(T_xX) \oplus T_zZ \text{ have positive orientation} \\ -1 & \text{otherwise} \end{cases}$$

   (a) If $X = \partial W$ and $f : X \to Y$ extends to W, then $I(f, Z) = 0$.
   (b) $I(f, g) = (-1)^{\dim(X) \cdot \dim(Z)} I(g, f)$.

(6) Transversality theorem. If $F : X \times S \to Y$ is smooth map (of manifolds!), only X has boundary, and $Z \subseteq Y$ is a boundaryless submanifold of Y, and $F, \partial F \pitchfork Z$, then for a.e. $s \in S$, we have both $f_s, \partial f_s \pitchfork Z$.

(7) Lefschetz fixed point theorem: Let $f : X \to X$ be smooth on a compact orientable manifold. If $L(f) \neq 0$, then f has a fixed point.
   (a) $L(f) = I(\Delta, \operatorname{graph}(f))$ where Δ is the diagonal.

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(b) $L(\text{id}) = I(\Delta, \Delta) = \chi(X)$, the Euler characteristic.
(c) A map is called *Lefschetz* if $\text{graph}(f) \pitchfork \Delta$. Every map is homotopic to a Lefschetz map. This means that if x is a fixed point, then df_x does not have 1 as an eigenvalue.
(d) $L(f) = \sum L_x(f)$ for fixed points x where

$$\begin{aligned} L_x(f) &= \begin{cases} +1 & \text{if } df_x - I \text{ preserves orientation on } T_x(X) \\ -1 & \text{if } df_x - I \text{ reverses orientation} \end{cases} \\ &= \deg\left(g : z \mapsto \frac{f(z) - z}{|f(z) - z|}\right) \end{aligned}$$

where the map $g : \partial B \to S^{k-1}$ on a small ball around the fixed point x. This definition works regardless of whether or not f is a Lefschetz map.

(8) Extension theorem: $Z \subseteq Y$ a compact submanifold (both $\partial Y = \partial Z = \varnothing$), $f : X \to Y$, $C \subseteq X$ closed. Suppose $f, \partial f \pitchfork Z$ on C and $C \cap \partial X$ respectively.
Then there is $g : X \to Y$ homotopic to f and identical on a neighborhood of C with $g, \partial g \pitchfork Z$ everywhere.
You can fix maps while preserving closed sets.
(9) Mod two intersection number and degree, self-intersection, and how to calculate
(10) $\mathbb{Z}$ intersection number and degree, self-intersection, and how to calculate
(11) Global flows. These are guaranteed by a compactly supported vector field or a compact manifold (here global includes existence for all time).
(12) Show that various matrix groups are manifolds.
Handy fact:

$$\frac{d}{dt} \det A(t) = \text{trace}(\text{adj}(A(t))\frac{dA(t)}{dt})$$

Also: $\det(I + tB) = 1 + t\,\text{tr}(B) + O(t^2)$ because λ eigenvalue of B implies $1 + t\lambda$ an eigenvalue of $I + tB$.
Better: $d\det_A B = \det A\,\text{tr}(A^{-1}B)$.

$$\frac{d}{dt}\det(A + tA) = \det A \frac{d}{dt}(1 + t)^n = \det A \cdot n \neq 0$$

so that for $A \in \text{SL}(n)$ or $A \in \text{GL}(n)$, the determinant is a submersion at A.
(13) Show that $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are manifolds.
A nice coordinate system is given on $U = \{x_0 \neq 0\}$ by $[1 : x_1 : \cdots : x_n] \to (x_1, \ldots, x_n)$. Transitions are smooth.
(14) A manifold as the preimage of a regular value. (Use the canonical form of submersions). Recall that $T_xM = \ker df_x$ if $x \in M = f^{-1}(y)$.
(15) Applications of partitions of unity (what things can you glue?)
(16) Poincaré-Hopf index theorem (to find the Euler characteristic of the sphere as well) only works on compact manifolds! Since every non-compact manifold has a non-vanishing vector field.
(17) Know the stereographic projection for spheres of arbitrary dimension. (not necessary. The result I had in mind, witness $\chi(S^n)$ using a vector field with Poincare-Hopf, does better with (subtract $e_n/4$ and then project back to sphere,

then take vector in this direction). The actual vector field isn't bad in (project on to hyperplane) coordinates, and neither is the diffeomorphism.

(18) Definitions of degree: either the number of pre-images of a regular value (equidimensional spaces), or the number n such that on top homology the map between top homologies (both $\mathbb{Z}$ assuming compact manifold) is $x \mapsto nx$.

(19) A proper local diffeomorphism from a connected space is a covering map (corollary of Ehresmann).

(20) Good counter-examples:
S^2 double cover of $\mathbb{RP}^2$, but $\pi_1(S^2)$ is zero, while $\pi_1(\mathbb{RP}^2)$ is not.
$(0, 1)$ cover (sans a point) of S^1 is local diffeo, but not covering map.

(21) A lie group has trivial tangent bundle (by carrying vectors from T_eM), so if it's compact, Poincare Hopf gives us Euler characteristic zero.

(22) Polar decomposition for $GL(n)$: $A = UP$ where U is unitary (orthogonal) and P is positive definite Hermitian (symmetric). Since $O(n)$ is a manifold (compact!) and the space of positive definite symmetric matrices is linear, the product is a manifold.

(23) Iwasawa decomposition for $SL(n)$: for $SL(n)$, we can decompose $G = KAN$ where $K = \mathrm{SO}(n)$, A is diagonal matrices whose entries are all positive and whose determinant is 1, and N is upper triangular matrices with 1s on the whole diagonal.

(24) Quotient manifold theorem: if G is a Lie group acting smoothly, freely (if $gx = x$, then $g = e$), and properly ($f : G \times X \to X$ is proper) on a manifold X, then G/X is a manifold with dimension $\dim X - \dim G$.

Special case, if G is a finite group: then a free action (necessarily smooth and proper since G is compact) can be quotiented by.

(25) Another version: if X is a set and G acts transitively such that for some point p, the isotropy group (stabilizier) G_p ($G_p = \{g \mid g \cdot p = p\}$) is closed in G, then X has a unique smooth manifold structure with respect to which the given action is smooth. Then $\dim X = \dim G - \dim G_p$.

As an example, let X be the set of k-dim subspaces of $\mathbb{R}^n$. Then $\mathrm{GL}(n)$ acts transitively on this space. Also the stabilizer / isotropy group of $\mathbb{R}^k \subseteq \mathbb{R}^n$ is the set of matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ where A, D are invertible, and B is any matrix. This set is closed in $\mathrm{GL}(n)$, so X is a manifold with dimension $\dim \mathrm{GL}(n) - \dim \mathrm{stabilizer}$.

(26) Grassmannian manifold (standard method): given subspace P of dimension k, for subspace V which intersects $P^\perp$ trivially, we write $V = \Gamma(X) = \{v + Xv\}$ for $X : P \to P^\perp$. Then Γ^{-1} gives a local Euclideanization (viewing linear maps as a Euclidean space). Consider P, P'. Let $\pi_P, \pi_{P'}$ be projections. Let $I_X : P \to \mathbb{R}^n$ be $I_X(v) = v + Xv$.

For S a subspace intersecting $P^\perp$ trivially, write $X' = (\pi_{P'^\perp} \restriction_S) \circ (\pi_{P'} \restriction_S)^{-1}$. Informally, given S, we get the linear map by (inverse) projecting down to P then projecting across to $P^\perp$.

To get X back, we write restriction to S as composition with I_X (whose image is S). Then X' depends smoothly on X.

(27) The space of lines in $\mathbb{R}^2$ is a line bundle over S^1 (the angle of the line) which is not trivial, so it's the Mobius bundle.

(28) Let $X = f^{-1}(0)$ be the pre-image of a regular value of $f : \mathbb{R}^n \to \mathbb{R}$. (Might need compactness of X). Then ∇f defines a normal vector field on X. Claim: $X \times S^1$ is parallelizable. Vector fields:

$$v_i(x, \theta) = (\pi_{T_x X} e_i, n \cdot e_i)$$

where $TS^1 \simeq S^1 \times \mathbb{R}$, but anything with a trivial bundle would work there. Linear independence is (more or less) easy.

(29) Borsuk Ulam Theorem:
   (a) $f : S^n \to \mathbb{R}^n$ continuous implies there exists x such that $f(-x) = f(x)$.
   (b) $f : S^n \to \mathbb{R}^n$ continuous and odd implies there exists a zero.
   (c) Corollary: Ham Sandwich. Pick translation of hyperplane to bisect the first set, then apply Borsuk Ulam to (vector $\mapsto$ volume in front of hyperplane for remaining sets).
   (d) $f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ continuous and odd implies $W_2(f, 0) = 1$.

(30) Whitney immersion, embedding theorems.
   (a) Every compact manifold can be embedded in $\mathbb{R}^N$ for some N. Look at finitely many charts ϕ_i with partition of unity β_i and then $f(p) = (\beta_1\phi_1(p), \ldots, \beta_k\phi_k(p), \beta_1(p), \ldots, \beta_k(p))$.
   (b) Any n-fold can be immersed (not necessarily injectively) in $\mathbb{R}^{2n}$. Consider $F(x, v) = df_x v$. Then $F : TM \to \mathbb{R}^N$, and Sard's theorem and dimension counting tells us that almost every value is not in the image. Take w not in the image. You can then project to $w^\perp$.
   (c) We can injective immerse into $\mathbb{R}^{2n+1}$ by considering also $H(x, y, t) = t(f(x) - f(y))$. Then a common regular value w is such that if we project to $w^\perp$, we stay an immersion, and furthermore, stay injective (since w is not in the image).

(31) Existence of a proper map. Cover X with charts ϕ_i with compact support. Write $f = \sum i\phi_i$.

(32) Why is graph($|x|$) not a smooth submanifold of $\mathbb{R}^2$?

Let $f : U \to \mathbb{R}^2$ be a chart about the origin such that f takes the graph to the x-axis.. Then f is a diffeomorphism, so in particular, $\nabla f_1, \nabla f_2 \neq 0$. But note that at the origin, $\nabla f_2 \cdot (1, 1) = 0$ and $\nabla f_2 \cdot (-1, 1) = 0$, so $\nabla f_2 = 0$. This is a contradiction!

Differential Geometry

(1) Lie bracket and Lie derivatives (with the flow interpretation)
   (a) We have $L_X f = Xf$
   (b) We define $(L_X\omega)(p) = \lim_{h\to 0} \frac{\phi_h^*\omega(p) - \omega(p)}{h}$.
   (c) We define $(L_X Y)(p) = \lim_{h\to 0} \frac{Y(p) - (\phi_h)_* Y(p)}{h}$. Keep in mind the order!

(2) Lie derivatives in coordinates. Let (x, U) be a coordinate system on M and suppose $X = \sum a^i \frac{\partial}{\partial x^i}$.

$$\begin{aligned} L_X(dx^i) &= \sum \frac{\partial a^i}{\partial x^j} dx^j \\ 0 = L_X\left(\delta^i_j\right) &= L_X\left(dx^i \frac{\partial}{\partial x^j}\right) \\ &= (L_X dx^i)\left(\frac{\partial}{\partial x^j}\right) + dx^i\left(L_X \frac{\partial}{\partial x^j}\right) \\ L_X \frac{\partial}{\partial x^j} &= -\sum \frac{\partial a^i}{\partial x^j}\frac{\partial}{\partial x^i} \end{aligned}$$

Conclusion:

$$\begin{aligned} L_X Y &= XY - YX \\ &= \sum_j \left(\sum_i a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x_i}\right)\frac{\partial}{\partial x^j} \end{aligned}$$

where $X = a^i X^i$ and $Y = b^i X^i$.

Also note that $L_Y(S \otimes T) = (L_Y S) \otimes T + S \otimes (L_Y T)$ for a vector field Y.

Another property:

$$L_X(T(Y_1, \ldots, Y_n)) = (L_X T)(Y_1, \ldots, Y_n) + T((L_X Y_1), \ldots, Y_n) + \cdots + T(Y_1, \ldots, (L_X Y_n))$$

(3) Writing a vector field in coordinates (it's $X = \sum a^i \frac{\partial}{\partial x^i}$ and then application of a function becomes $X(f) = \sum a^i \frac{\partial f}{\partial x^i}$)

(4) Exterior derivative in coordinates and other formulae for it.

For a form $\omega = g_I dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, we have

$$d\omega = dg_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

(5) If $[X, Y] = 0$, then the flows commute for all time.

(6) Frobenius Theorem and equivalent statements about differential forms and wedges

Vector field facts: If $X_1, \ldots, X_k$ are linearly independent around p and their commutators vanish, then there is a coordinate system (x, U) around p such that $X_\alpha = \frac{\partial}{\partial x^\alpha}$ for $\alpha = 1, \ldots, k$.

To prove this, look at flows ϕ^α_t for X_α. Make sure $X_\alpha(0) = \frac{\partial}{\partial t^\alpha}$ at the origin. Then define

$$\chi(a^1, \ldots, a^n) = \phi^1_{a_1}(\phi^2_{a^2}(, \ldots, (\phi^k_{a^k}(0, 0, \ldots, 0, a^{k+1}, \ldots, a^n)) \ldots)).$$

Using $x = \chi^{-1}$ as coordinates, we get that $X_1 = \frac{\partial}{\partial x^1}$. Commutativity lets us interchange the order of flows, since they commute.

For a 1-form ω on a (≥ 3)-manifold, $\ker \omega$ is integrable iff $\omega \wedge d\omega = 0$. Proof: use $dI(\Delta) \subseteq I(\Delta)$ for one direction and $d\omega(x, y) = \omega(x) - \omega(y) - \omega([x, y])$ for the other direction with $x, y \in \ker \omega \implies [x, y] \in \ker \omega$. This holds for a 1-form on any manifold (of dimension 3 or higher) with roughly the same proof.

(7) Boundary orientation $\{n, v_1, \ldots, v_{k-1}\}$ is oriented in M iff $\{v_1, \ldots, v_{k-1}\}$ is oriented in ∂M. *Outward normal first!*

(8) Stokes' Theorem and its classical interpretations.

$$\int_{\partial M} \omega = \int_M d\omega$$

where $\int_{\partial M} \omega$ is done by taking $i : \partial M \to M$ and then integrating $i^*\omega$. We define the orientation for ∂M by putting the outward normal first.

Interpretations include divergence theorem ($\int_{\partial M} V \cdot nds = \int_M \operatorname{div} V\, d\,\mathrm{vol}$) and Stokes theorem (classical) $\int_\gamma v \cdot Tdt = \int_S \operatorname{curl} v \cdot ds$.

For the divergence theorem, we define $\omega = V_1(dx^2 \wedge \cdots \wedge dx^n) + V_2(dx^1 \wedge dx^3 \wedge \cdots \wedge dx^n) + \dots$. Then $d\omega = \operatorname{div} V\, d\mathrm{vol}$. We note that $i^*_{\partial M}(i_n\omega) = n \cdot V ds$, where ds is the area form, and n is the outward normal. This is by writing locally $dvol = dn \wedge ds$ where here n is a function.

(9) Cartan Magic Formula with proof

The formula is $L_X\omega = i_X d\omega + d(i_X\omega)$. Proof: for 0–forms it's trivial. Induct.

$$\begin{aligned}
\omega &= \sum f_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
&= d(x^{i_1}) \wedge f_I dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\
&= da \wedge b
\end{aligned}$$

Calculate both sides, and using product rule, linearity, things work out.

(10) The form $dz - ydx$ has nowhere integrable kernel.

(11) $d\omega(x, y) = \omega(x) - \omega(y) - \omega([x, y])$ for a 1-form ω.

(12) If $f \simeq g : M \to N$ are homotopic, then $f^*\omega$ and $g^*\omega$ are cohomologous if ω is closed.

Let $F : M \times [0, 1] \to N$ be such that $F_0 = f$ and $F_1 = g$. Then

$$\begin{aligned}
(f_* - g_*)(\omega) &= \int_0^1 L_{\frac{\partial}{\partial t}} \omega \\
&= \int_0^1 i_t d\omega + di_t\omega \\
&= \left(\int_0^1 i_t \cdot\right) \circ d(\omega) + d \circ \left(\int_0^1 i_t \cdot\right) \omega \\
&= hd\omega + dh\omega \\
&= d(h\omega)
\end{aligned}$$

Thus $f^*\omega$ and $g^*\omega$ differ by a closed form.

(13) Poincaré Duality: for a connected oriented n–dim manifold ("of finite type"), the map $PD : H^k(M) \to H_c^{n-k}(M)^*$ is an isomorphism for all k. The map is given by

$$PD(\omega)(\alpha) = \int \omega \smile \alpha$$

since $H^n_c(M)$ is $\mathbb{R}$ via the isomorphism given by integration.

In particular, if M is compact, $H^k \to H^{n-k}(M)^*$ is an iso.

(14) de Rham's theorem: we know that $H^k(M; \mathbb{R}) \simeq \operatorname{Hom}(H_k(M; \mathbb{R}); \mathbb{R})$, but in fact for $\alpha : \Delta^k \to M$ a chain we can integrate a differential form on it (wlog it's

smooth (up to homotopy)), and so we have an isomorphism of vector spaces (from de Rham cohomology to singular):

$$\omega \mapsto (\alpha \mapsto \int_\alpha \omega)$$

where the image of ω is considered an element of H^k in singular cohomology. De Rham's theorem is the statement that this is an isomorphism of vector spaces (so singular and de rham cohomology are the same).

(15) Poincaré lemma. This is the lemma that says on a contractible space, closed and exact forms are the same.
(16) A closed form is exact on a sphere if and only if the integral is zero, since $\int \cdot : H^{\text{top}} \to \mathbb{R}$ is an isomorphism by Poincaré duality.
(17) a chain homotopy: a map $h : A_{n-1} \to B_n$ such that $hd + dh = f - g$ (where f, g are maps from the A chain complex to the B chain complex).
(18) Thom Class: Let M be a compact connected oriented manifold, $\pi : E \to M$ an oriented k-dim vector bundle with orientation ν. Then the *Thom class* U is the unique element of $H_c^k(E)$ with the property that for all $p \in M$, we have $j_p^* U = \nu_p$, i.e.

$$\int_{F_p, \nu_p} j_p^* \omega = 1$$

where U is the class of the closed form ω and $j_p : F_p \to E$ is the inclusion of the fiber into the space.

Alternatively: it's the U such that $\pi^*\mu \smile U = \mu \oplus \nu \in H_c^{n+k}(E)$, where μ is the top class on M (basically the orientation).

Algebraic Topology

(1) Smash, join, wedge sum, connect sum
  (a) Wedge: $A \vee B$ where A is pointed with p and B pointed with q is

$$A \vee B = (\{p\} \times B) \cup (A \times \{q\}) \subseteq A \times B$$

  (b) Join: $X * Y$ is

$$X * Y = X \times Y \times I / \sim$$

  where $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$, so it collapses $X \times Y \times \{0\}$ to X and on the other end to Y.
  (c) Smash: $X \wedge Y = X \times Y / X \vee Y$.
  (d) Connect sum: remove disks from X and Y and glue along a map from $S^k \to S^k$. Note that every compact surface is the connect sum of spheres, projective planes, and tori (uniquely (except for the sphere)).
(2) Homotopy extension property for a pair (X, A) is: given a map $f_0 : X \to Y$ then an agreeing homotopy $f_t : A \to Y$ extends to a homotopy $f_t : X \to Y$. This holds for a CW pair.
(3) A space is contractible iff every map into it is nullhomotopic iff every map out of it is nullhomotopic.
(4) Given (X, A) where X retracts to A, the map induced by the inclusion on $\pi_1(A) \to \pi_1(X)$ is injective. (Corollary: Brouwer Fixed Point).

(5) Van Kampen: if X is the union of path-connected open sets A_α each containing the same basepoint, and if each intersection is path-connected, then $\phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective.

If each triple intersection is path-connected, then we know the kernel and get an isomorphism

$$\pi_1(X) \simeq *_\alpha \pi_1(A_\alpha)/N$$

with$N = \langle i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} \rangle$.

Basically, make equivalent the different ways of including things from intersections into the A_α and A_β.

(6) For cell complexes, π_1 is generated by the 1-cells and quotiented by relations generated by considering each boundary of each 2-cell. For instance, the torus has a cell complex structure given by one vertex v, two loops a, b, and a two-cell f attached by taking a, b, a^{-1}, b^{-1} to be its boundary. Then $\pi_1(T^2) = \langle a, b \mid [a, b] \rangle$.

(7) Covering spaces and π_1.

A covering map is $p : \tilde{X} \to X$ such that there's an open cover $\{U_\alpha\}$ of X with $p^{-1}(U_\alpha)$ a disjoint union of open sets, homeomorphic via p to U_α. NB: p need not be surjective.

Fact: the induced map $p_* : \pi_1(\tilde{X}) \to \pi_1(X)$ is always injective.

Changing the basepoint selection for $\tilde{X}$ produces a subgroup conjugate to $p_*(\pi_1(\tilde{X}))$ in $\pi_1(X)$, with conjugating element represented by the projection of the path from old basepoint $\tilde{x}_0$ to new basepoint $\tilde{x}_1$.

In this way, there is a 1-1 correspondence between covering spaces $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and actual subgroups of $\pi_1(X, x_0)$. By forgetting basepoints, we get a 1-1 correspondance between covering spaces $p : \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X)$.

(8) Universal cover: the covering space corresponding to the trivial subgroup of $\pi_1(X)$.

(9) Lifting of maps: given $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$, and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, there exists a unique homotopy $\tilde{f}_t$ lifting f_0.

(10) The number of sheets of a covering space (if both X and $\tilde{X}$ are path connected) is the index of $p_*(\pi_1(\tilde{X}))$ in $\pi_1(X)$.

(11) *Path lifting criterion:* A lift exists iff $f_*(\pi_1(Y)) \subseteq p_*(\pi_1(\tilde{X}))$.

(12) *Unique lifting property:* given two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ of $f : Y \to X$, if Y is connected and the lifts agree at a point, then they agree everywhere.

(13) Isomorphisms of a covering map $\tilde{X} \to \tilde{X}$ are called deck transformations. They form a group $G(\tilde{X})$. A covering space is *normal (or regular)* if for every $\tilde{x}, \tilde{x}'$ above every x, $\exists g \in G.\ g\tilde{x} = \tilde{x}'$.

A covering space is normal iff the subgroup it corresponds to is normal.

$$G(\tilde{X}) \cong N(H)/H$$

where H is the subgroup $p_*(\pi_1(\tilde{X}))$, and $N(H)$ is the normalizer of H in $\pi_1(X)$.

Reminder: $N_G(S) = \{g \in G \mid gS = Sg\}$ is the normalizer of S in G.

(14) For an entirely discontinuous (here, preimages of open sets are disjoint) action of G on Y, we get a normal covering space $Y \to Y/G$ whose deck transformation group is G. Also $G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$ if Y is path-con and locally path-con.

(15) The action of the deck transformation group is not the same as the action of $\pi_1(X)$ via lifting loops. One is a right action and the other is a left action.
(16) Fundamental group facts
   (a) $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$ when X, Y are path connected.
(17) Fundamental group of complement of linked circles: the complement deformation retracts to $S^2 \vee T^2$.
(18) (X, A) is a good pair if a neighborhood of A deformation retracts to A.
(19) Various long exact sequences
   (a) X is a space and A a non-empty closed subspace with (X, A) a good pair (i.e., a neighborhood of A def rets to A.). Then

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \cdots \to \tilde{H}_0(X/A) \to 0$$

   where i is the inclusion, j the quotient map, and ∂ the map induced by the snake lemma.
   (b) For a pair (X, A) we have the relative homology long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots \to H_0(X, A) \to 0$$

   The map ∂ takes a class α (represented by a chain x) in $H_n(X, A)$. We view x as a chain on X (unique up to a boundary), and then take the boundary ∂x which defines a class in $H_{n-1}(A)$. It's diagram chasing.
   A better description: let $[\alpha] \in H_n(X, A)$ be a class represented by a relative cycle. Then $\partial[\alpha]$ is the class of a cycle $\partial\alpha$ in $H_{n-1}(A)$. Note that a relative cycle is an element of the quotient $C_k(X)/C_k(A)$.
   (c) A triple (X, A, B) yields the SES

$$0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0$$

   and hence the LES

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to \cdots \to H_0(X, A) \to 0$$

   (d)
(20) Excision Theorem: Given $Z \subseteq A \subseteq X$ such that $\bar{Z} \subseteq A^\circ$, then $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms on all homology.
(21) Five lemma: for $A \to B \to \cdots \to E$ forming a rectangle with $A' \to B' \to \cdots \to E'$, if the two rows are exact, and all but the center are isomorphisms, then the center is an isomorphism too. Label the descending maps $\alpha, \beta, \gamma, \delta, \varepsilon$. This follows then by the Four Lemma: if β, δ surjective and ε injective, then γ is surjective. (And the whole thing dualized, too.)
(22) Suspension preserves degree: $\deg Sf = \deg f$ where Sf is the induced map on suspension.
(23) $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$
(24) $\chi(\mathbb{RP}^2) = 1$ (half of the corresponding sphere (0 or 2 depending on parity of dimension)).
(25) Relative homology: Given $A \subseteq X$, we get $i : A \to X$ inducing $C_p(A) \to C_p(X)$, so we can quotient, and get $C_p(X, A) = C_p(X)/C_p(A)$. We get a chain complex by the boundary map on $C(X)$ (ends up being well defined). Can take homology. SES of chains gives us LES on homology (snake lemma).
(26) Snake lemma: SES yields LES with diagram chasing.

(27) Mayer-Vietoris: for a pair (X, A)

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to \cdots \to H_0(X) \to 0$$

Here, the first map is induced by $(x, -x)$ and the second map by $(x, y) \mapsto (x + y)$. The boundary map takes $\alpha \in H_n(X)$ represented by a cycle z and by choosing $z = x + y$ a sum of chains in A and B, we get that $\partial x = -\partial y$ (since $\partial(x + y) = 0$), so $\partial\alpha \in H_{n-1}(A \cap B)$ is represented by $\partial x = -\partial y$.

(28) Boundary map on product of CW complexes (how to calculate)

$$d(ef) = d(e)f + (-1)^{\text{dimension of cell e}} ed(f)$$

(29) Cell decompositions of all surfaces:
   (a) Sphere: e_2 attached via the constant map to e_0.
   (b) Genus g surface: take a $4g$-gon, and label the edges $a, b, \bar{a}, \bar{b}, c, d, \bar{c}, \bar{d}, \ldots$. Attach the single two-cell.
   (c) $\mathbb{RP}^2$: square with all opposite sides identified opposite-wise.
   (d) Klein bottle: Mobius strip turned into a cylinder.
   (e) Surface with g copies of $\mathbb{RP}^2$ connect summed together (crosscap number g) has a $4g$-gon with edges labelled $a, a, b, b, c, c, d, d, \ldots$.

(30) Mapping torus example: let $f, g : X \to Y$ and define $Z = X \times I \sqcup Y/\!\sim$ where

$$(x, 0) \sim f(x) \in Y$$
$$(x, 1) \sim g(x) \in Y$$

Consider $q : (X \times I, X \times \partial I) \to (Z, Y)$ the restriction to $X \times I$ of the quotient map $X \times I \sqcup Y \to Z$. We get an induced map on long exact sequences of relative homology.

Top row: $\to H_{n+1}(X \times I, X \times \partial I) \to H_n(X \times \partial I) \to H_n(X \times I) \to \ldots$.

Note that the third map is surjective, so the first and last maps are zero. Thus the second map is injective. Since the middle thing is $H_n(X) \oplus H_n(X)$, and the last thing is $H_n(X)$, this means that the first group is also $H_n(X)$.

Bottom row: $\to H_{n+1}(Y, Z) \to H_n(Y) \to H_n(Z) \to \ldots$.

Since q is a homeomorphism from $(X \times I, X \times \partial I) \to (Z, Y)$, the q_* from $H_{n+1}(X \times I, X \times \partial I) \to H_{n+1}(Y, Z)$ is an isomorphism.

The result is

$$\cdots \to H_n(X) \xrightarrow{f_* - g_*} H_n(Y) \xrightarrow{i_*} H_n(Z) \to H_{n-1}(X) \to \ldots$$

(31) The mapping torus: if $X = Y$ and $f = \text{id} : X \to X$, then Z is the mapping torus.

We get the exact sequence

$$\cdots \to H_n(X) \xrightarrow{\text{id} - g_*} H_n(X) \xrightarrow{i_*} H_n(Z) \to \ldots$$

(32) The nth Betti number is defined to be the rank of H_n.
(33) Compute relative (cellular) homology
(34) Homology of $\mathbb{CP}^n$ is alternating $0, \mathbb{Z}$ (starting at $\mathbb{Z}$ naturally), since one cell in each even dimension.
(35) Homology of common spaces:
   (a) $H_*(\mathbb{CP}^n; \mathbb{Z})$ is $0, \mathbb{Z}, 0, \mathbb{Z}, \ldots$. This is because it's equal to the cellular chain complex.

(b) $H_*(\mathbb{RP}^n;\mathbb{Z})$ is $\mathbb{Z},\mathbb{Z}_2,0,\mathbb{Z}_2,\dots$ and ends on $\mathbb{Z}$ if n odd and 0 otherwise (consider orientability).
(c) $H_*(\mathbb{RP}^n;\mathbb{Z}_2)$ is $\mathbb{Z}_2$ across the board.
(d) $H_*(S^n;\mathbb{Z})$ is $\mathbb{Z}$ in 0 and n.
(e) $H_*(T^2;\mathbb{Z})$ is $\mathbb{Z},\mathbb{Z}^2,\mathbb{Z}$.
(f) $H_*(\Sigma_g;\mathbb{Z})$ where Σ_g is the surface of genus g is $\mathbb{Z},\mathbb{Z}^{2g},\mathbb{Z}$ (so $\chi(\Sigma_g)=2-2g$).
(g) $H_*(K;\mathbb{Z})$ is $\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_2,0$ in increasing order (using the mapping torus sequence with g equal to the antipodal map, or glue two Möbius strips and apply Mayer-Vietoris.
(h) $H_k((S^1)^n;\mathbb{Z})=\mathbb{Z}^{\binom{n}{k}}$. This follows from the Kunneth formula for homology (which is complicated), or complicated cellular homology calculation. If you want it with $\mathbb{R}$ coefficients, use UCT and Kunneth on cohomology.
(i) $H_*(M)=H_*(S^1)$ since the Möbius band deformation retracts to the (center) circle.

(36) Universal Coefficient Theorem for homology: if the chain group consists of free abelian groups, then the following are (not-naturally) split exact sequences.

$$0\to H_n(C)\otimes G\to H_n(C;G)\to \mathrm{Tor}(H_{n-1}(C),G)\to 0$$

where
(a) $\mathrm{Tor}(A,B)=\mathrm{Tor}(B,A)$
(b) $\mathrm{Tor}(\oplus_i A_i,B)=\oplus_i \mathrm{Tor}(A_i,B)$
(c) $\mathrm{Tor}(A,B)=0$ if either are free (or torsionfree more generally).
(d) $\mathrm{Tor}(A,B)=\mathrm{Tor}(T(A),B)$ where $T(A)$ is the torsion subgroup.
(e) $\mathrm{Tor}(\mathbb{Z}_n,A)=\ker(A\xrightarrow{n}A)$

(37) Universal Coefficient Theorem for cohomology: The following is a split exact sequence.

$$0\to \mathrm{Ext}(H_{n-1}(C),G)\to H^n(C;G)\to \mathrm{Hom}(H_n(C),G)\to 0$$

Facts about the Ext groups:
(a) $\mathrm{Ext}(H\otimes H',G)=\mathrm{Ext}(H,G)\otimes\mathrm{Ext}(H',G)$
(b) $\mathrm{Ext}(H,G)=0$ if H is free.
(c) $\mathrm{Ext}(\mathbb{Z}_n,G)=G/nG$.
so in particular, for $G=\mathbb{R}$ or really any field, the Ext groups are trivial, so $H^n=\mathrm{Hom}(H_n)$.

Corollary: if H_n,H_{n-1} are finitely generated with torsion T_{n-1},T_n, then

$$H^n(C;\mathbb{Z})\cong (H_n/T_n)\oplus T_{n-1}$$

so for $\mathbb{Z}$ coefficients, torsion moves up.

Useful commutative diagram: top row: $0\to\mathrm{Ext}(H_{n-1}(C),G)\to H^n(C;G)\to \mathrm{Hom}(H_n(C),G)\to 0$. Bottom row has C'. Vertical maps are $(\alpha_*)^*$, α^*, and $(\alpha_*)^*$. In particular, if Ext is trivial, then the map on cohomology is the dual of the map on homology.

(38) Mayer-Vietoris for cohomology:
(a) Absolute Mayer-Vietoris

$$\cdots\to H^n(X;G)\to H^n(A;G)\oplus H^n(B;G)\to H^n(A\cap B;G)\to H^{n+1}(X;G)\to\dots$$

(b) Relative Mayer-Vietoris

$$\cdots \to H^n(X;Y) \to H^n(A,C) \oplus H^n(B,D) \to H^n(A\cap B, C\cap D) \to \dots$$

where $X, Y) = (A \cup B, C \cup D)$ with $C \subseteq A$, and $D \subseteq B$ such that X is the union of the interiors of A, B and Y the union of the interiors C, D.

(39) Cohomology Rings:
(a) $H^*(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ where $|\alpha| = 1$
(b) $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$ where $|\alpha| = 1$
(c) $H^*(\mathbb{RP}^{2k}; \mathbb{Z}) = \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1})$ where $|\alpha| = 2$
(d) $H^*(\mathbb{RP}^{2k+1}; \mathbb{Z}) = \mathbb{Z}[\alpha, \beta], (2\alpha, a^{k+1}, \beta^2, \alpha\beta)$ where $|\alpha| = 2, |\beta| = 2k+1$.
(e) $H^*(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$ where $|\alpha| = 2$
(f) $H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[\alpha]$ where $|\alpha| = 2$.
(g) $H^*(\mathbb{HP}^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$ where $|\alpha| = 4$
(h) $H^*(\mathbb{HP}^\infty; \mathbb{Z}) = \mathbb{Z}[\alpha]$ where $|\alpha| = 4$.
(i) $H^*(T^n; \mathbb{Z}) = \Lambda_{\mathbb{Z}}[\alpha_1, \dots, \alpha_n]$ with the monomial $\alpha_{i_1} \dots \alpha_{i_k}$ corresponding to the cell $e^1_{i_1} \times \dots \times e^1_{i_k}$.
(j) $H^*(S^k; \mathbb{Z})$ is either $\Lambda[\alpha]$ or $\mathbb{Z}[\alpha]/(\alpha^2)$. These are isomorphic as rings, but in the graded sense their tensors differ. The former case happens if k is odd, and the latter if k is even, and α has odd or even grading correspondingly.
(k) $H^*(S^{k_1} \times \dots \times S^{k_n}; \mathbb{Z}) = \Lambda_z[\alpha_1, \dots, \alpha_n]$ if each k_i is odd, since $\Lambda[\alpha_1, \dots, \alpha_n]$ is the tensor of $\Lambda[\alpha_i]$. Similarly it becomes $\mathbb{Z}[\alpha_1, \dots, \alpha_n]/(\alpha_1^2, \dots, \alpha_n^2)$ if each k_i is even.
(l) $H^*(X; Z_p) = H^*(X; \mathbb{Z}) \otimes \mathbb{Z}_p$ as rings if $H_n(X; \mathbb{Z})$ is finitely generated and free for each n.

(40) Kunneth Formula

$$H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y; R)$$

is an isomorphism of rings if X, Y are CW complexes and $H^k(Y;R)$ is finitely generated and free for all k.

(41) Facts about tensors of graded rings:
(a) In general, $R[\alpha_1, \dots, \alpha_m] \otimes R[\beta_1, \dots, \beta_n]$ is $R[\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n]$.
(b) $\mathbb{R}^m \otimes \mathbb{R}^n = \mathbb{R}^{mn}$ (in particular, $\mathbb{R} \otimes 0 = 0$)
(c) $\mathbb{Z}_m \otimes \mathbb{Z}_n = \mathbb{Z}_{\gcd(m,n)}$.
(d) To find the specific groups:

$$H_n(X \times Y) = \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$$

(42) If M is a compact connected n-manifold, then $H_{n-1}(M;\mathbb{Z})$ has no torsion if M is orientable. Otherwise has $\mathbb{Z}_2$ torsion if non-orientable.

(43) For M connected *non-compact* n-fold, $H_i = 0$ for $i > n$ and $i = n$.

(44) Poincare duality: for a closed R-orientable n-fold with fundamental class $[M] \in H_n(M;R)$, the map

$$D : H^k(M;R) \to H_{n-k}(M;R)$$

defined by $D(\alpha) = [M] \frown \alpha$ is an isomorphism. Recall that the cap product $\frown$ is given by contraction: for $\sigma : \Delta^p \to X$ a chain and $z \in C^q(X;R)$ a chochain, we get

$$\sigma \frown z = z(\sigma\restriction_{[v_0,\dots,v_q]})\sigma\restriction_{[v_q,\dots,v_p]}$$

An alternative version is given in Spivak. Let M be a connected oriented n-manifold "of finite type". Then

$$PD : H^k(M) \to H_c^{n-k}(M)^*$$
$$PD(\alpha)(\beta) = \mu(\alpha \cup \beta)$$

is an isomorphism for all k. Here we use $\mathbb{R}$ coefficients and μ is integration against a volume form (with integral 1).

(45) Note: the pairing $(\alpha, \beta) \mapsto (\alpha \cup \beta)[M]$ is always a non-degenerate bilinear pairing for a compact orientable manifold M.

(46) Induced maps on cohomology rings.
   (a) All maps from $S^2 \to S^1 \times S^1$ have degree zero. On one hand this can be seen by π_1, but also the cohomology ring of T^2 is $\mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ so the map on H^1 is trivial, and hence the map on cohomology rings is trivial.

(47) Top homology of compact manifold (w/o boundary) is $\mathbb{Z}$, but compact manifold with boundary yields 0.

(48) Odd dimensional compact orientable manifolds have euler characteristic zero. This is a corollary of Poincare Duality, since the H^k and H^{n-k} terms cancel in pairs.

(49) Poincaré Duality tricks: let M be a compact $(4n + 2)$-manifold. Show that H^{2n+1} is even dimensional ($\mathbb{R}$ coefficients). It's enough to note that $\frown$ is a non-degenerate (by Poincaré Duality, every non-trivial form has another it can cup with to get 1 in top cohomology) alternating bilinear form. By writing it as a matrix, we get $\det(A) = \det(A^\top) = \det(-A) = (-1)^{\dim H^{2n+1}} \det(A)$. Thus it's even dimensional.

(50) Nice deformation retractions:
   (a) $\mathbb{R}^3 \setminus$ (two linked circles) deformation retracts to $S^2 \vee T^2$.
   (b) $\mathbb{R}^3 \setminus$ (two unlinked circles) deformation retracts to $S^2 \vee S^1 \vee S^2 \vee S^2$.
   (c) $\mathbb{R}^n \setminus$ (axes) deformation retracts to $S^{n-1} \setminus (2n$ points) which def retracts to $D^{n-1} \setminus (2n - 1$ points) and then $\bigvee_{2n-1} S^{n-2}$
   (d) $\mathbb{R}^3 \setminus$ (circle) deformation retracts to S^2 with a diameter, i.e., $S^2 \vee S^1$.
   (e) A torus without k points deformation retracts to $\bigvee_{k+1} S^1$.

Extra Topics

(1) Connect sums: calculation (use Van Kampen or Mayer-Vietoris).
(2) Concrete computation of H_{dR} of S^1, and S^2 (using Poincaré Lemma).
(3) Homology of $CX, SX, \Sigma X$.
(4) More matrix groups and why they're manifolds: $O(n), \mathrm{SL}(n), \mathrm{GL}(n)$. Also matrices of rank r.
(5) Proof of the Ehresmann fibration theorem.